

Capillary water waves in a canal with perpendicular contact angle

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Abstract

1 Introduction

The water waves problem studies the motion of a fluid in a container, separated from the atmosphere by a free-moving interface, under the action of gravity and surface tension. Here we consider the cases of an infinite canal and of a rectangular basin, with vertical walls.

More precisely, the fluid occupies a time-dependent region Ω_t situated below a moving surface Σ_t . Thus the vertical — up-pointing — direction e_y is distinguished from the horizontal directions e_{x_1}, e_{x_2} . To simplify matters we assume that Σ_t is the graph of a function $\eta(t, x_1, x_2)$ — this means that we forbid configurations where the fluid overhangs itself, like rolls or breakers. Thus for $t \in [0, T]$,

$$\Omega_t := \{(x_1, x_2, y) \in M \times \mathbf{R}; b(x) < y < \eta(t, x), x := (x_1, x_2)\},$$

where $M = (0, l) \times \mathbf{R}$ in the case of an infinite canal or $M = (0, l) \times (0, L)$ in the case of a rectangular basin, and where b is a real continuous function on M , describing the topography of the bottom. The function η is real, continuous, and defined on $[0, T] \times M$. We define by Γ_t the portion of the fixed boundary that is underwater,

$$\Gamma_t := \partial\Omega_t \setminus \Sigma_t.$$

An important assumption on the domain is that the surface only meets the container along the vertical walls. Mathematically this means that for $t \in [0, T]$,

$$(B) \quad \exists h_t > 0, b(x) < \eta(t, x) - h_t, \forall x \in M.$$

The fluid is assumed to be perfect, incompressible, non-viscous, and of constant density and temperature. Thus its velocity field $u(t, x_1, x_2, y) \in \mathbf{R}^3$, defined for $t \in [0, T], (x_1, x_2, y) \in \overline{\Omega}_t$, follows the incompressible Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla_{x,y} u + \nabla_{x,y} P = -ge_y, \\ \nabla_{x,y} \cdot u = 0, \end{cases}$$

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where $x := (x_1, x_2)$ stands for the horizontal variables, $P(t, x, y) \in \mathbf{R}$ is the pressure, and $g \geq 0$ is the acceleration of gravity, supposed uniform and constant. To simplify the study and focus on the dynamics of the surface, it is customary to impose in addition the condition

$$\operatorname{curl}_{x,y} v = 0,$$

which is conserved by the flow.

The free surface is assumed to move with the fluid velocity, thus

$$\partial_t \eta(t, x) = \nu(t, x) \cdot u(t, x, \eta(t, x)) = u_y(t, x, \eta(t, x)) - \nabla \eta(t, x) \cdot u_x(t, x, \eta(t, x))$$

where $\nu = (-\nabla \eta, 1)$ is the exterior normal to Σ_t , and $u := (u_x, u_y)$, with $u_x := (u_{x_1}, u_{x_2})$. The velocity field value at the boundary Γ_t of the container needs to satisfy the no-penetration conditions

$$u \cdot n = 0$$

where n is the normal to Γ_t . This and the equations prescribe the normal derivative of the pressure at Γ_t , thus we need only to prescribe the pressure at the free surface Σ_t . The presence of surface tension corresponds to a jump of the pressure across the interface, proportional to its mean curvature

$$H(\eta) := \operatorname{div} \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right).$$

Assuming the atmospheric pressure to be a constant — which can be normalized to 0 since the equation depends only on its gradient — this means that

$$P(t, x, \eta(t, x)) = -\kappa H(\eta)(t, x),$$

where $\kappa \geq 0$ is the surface tension coefficient.

Under the incompressibility and irrotationality conditions, the velocity of the fluid is the gradient of a harmonic function $\phi : \Omega_t \rightarrow \mathbf{R}$. Thus

$$v = \nabla_{x,y} \phi,$$

with

$$(1.1) \quad \Delta_{x,y} \phi = 0 \quad \text{in } \Omega_t.$$

The boundary conditions on ϕ translate as

$$(1.2) \quad \partial_n \phi = 0 \quad \text{on } \Gamma_t.$$

Thus if we know the value of ϕ at the free surface Σ_t , we can solve the Laplace problem (1.1) in Ω_t with this Dirichlet datum at Σ_t and Neumann condition at the rest of the boundary $\partial\Omega_t$. Thus we only need to know the evolution of the function

$$\psi(t, x) := \phi(t, x, \eta(t, x)).$$

From the Euler equation and the Pressure value at the surface, we see that

$$\partial_t \phi = -g\eta + \kappa H(\eta) - \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} (\partial_y \phi)^2 \quad \text{on } \Sigma_t,$$

while the dynamic boundary condition becomes

$$\partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi \quad \text{on } \Sigma_t.$$

Thus, by introducing the (rescaled) Dirichlet-Neumann map

$$(G(\eta)\psi)(t, x) = \partial_y \phi(t, x, \eta(t, x)) - \nabla \eta(t, x) \cdot \nabla \phi(t, x, \eta(t, x)) = \sqrt{1 + |\nabla \eta|^2} \partial_n \phi|_{y=\eta(t, x)},$$

we obtain the system

$$(1.3) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - \kappa H(\eta) + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0, \end{cases}$$

which is due to Zakharov ([17]) and Craig-Sulem ([10]). Here the advantage is that η and ψ are both real-valued functions of $(t, x) \in [0, T] \times M$.

In this case, the surface Σ_t and the container Γ intersect, and the exact nature of the boundary conditions to be imposed at this intersection are unknown. The Cauchy problem has been extensively studied in the case where Σ_t and Γ are clearly separated, corresponding to a laterally infinite ocean with a fixed separation between the bottom and the free surface — see for instance Nalimov ([11]), Shinbrot ([12]), Yosihara ([16, 15]), Craig ([9]), Beyer and Günther ([6]), and Wu ([13, 14]), and the works of Alazard, Burq, and Zuily ([1, 3, 4, 5]). However, if this separation condition is removed, almost no such results exist. One exception, when surface tension is absent, is in the case described above of an infinite canal or a rectangular basin with vertical walls. In this setting, local existence has been proved in [2] using the observation that the angle between the free surface and the walls is necessarily a right angle, and simple reflection and periodisation tricks to deduce local existence from results valid on the Torus.

In this article, we prove a similar result in presence of surface tension, thus $\kappa \neq 0$, and we normalize it to 1. Here, there is no physical obligation for the surface to meet the boundary at a right angle, however we will see that if such is the case initially, it remains true for a short time. To encode this information, we will introduce for $s > 1 + \frac{d}{2}$ the space

$$H_p^s(M) := \{u \in H^s(M); \partial_\nu u = 0 \text{ on } \partial M\},$$

where ν is the normal to ∂M . In the case of the rectangular basin, the normal is not defined at the corners, but this means that both $\partial_{x_1} u$ and $\partial_{x_2} u$ are 0 there. Since $s > 1 + \frac{d}{2}$, $H^s(M) \subset C^1(\overline{M})$ so that the definition makes sense. This corresponds to Sobolev functions that meet the walls of the container at right angles. Our theorem is as follows.

Theorem 1.1. *Take $s \in (3 - \frac{3}{10}, 3)$, $2 < r < s + \frac{3}{10} - 1$, and $M = (0, l) \times \mathbf{R}$ or $M = (0, l) \times (0, L)$. Consider initial data*

$$(\eta_0, \psi_0) \in H_p^{s+\frac{1}{2}}(M) \times H_p^s(M),$$

satisfying Assumption (B) at initial time $t = 0$. Then there exists a time $T > 0$ depending only on the norms of those initial data and the h_0 in Assumption (B) such that the Cauchy Problem of (1.3) admits a unique solution

$$(\eta, \psi) \in C \left([0, T]; H_p^{s+\frac{1}{2}}(M) \times H_p^s(M) \right) \cap L^2 \left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M}) \right),$$

satisfying Assumption (B) for each time $t \in [0, T]$.

Remark. 1. The condition $s < 3$ is a necessary limitation to the method. Even starting from smooth initial data, the symmetrization trick will transform η to an $H^{s+\frac{1}{2}}$ function of the Torus, with $s < 3$, so that the solution we construct will only have this regularity. See Section 2 for details.

2. To solve the Cauchy problem at this low regularity, the energy estimates are insufficient, and one need to use Strichartz estimates for this equation on the Torus. Nguyen and the author have proved such Strichartz estimates and used them to solve the Cauchy Problem, in the case of the whole space, in [?, ?]. We explain in Section 3 how this adapt to the Torus.
3. The use of such Strichartz estimates explains why the uniqueness is only known to holds in the space

$$C\left([0, T]; H_p^{s+\frac{1}{2}}(M) \times H_p^s(M)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right).$$

4. In [2], Alazard, Burq and Zuily proved the corresponding result in the absence of surface tension, within the class of uniformly local Sobolev spaces. In the case of the infinite canal, this means that no decay of the data at infinity is required. However in the presence of surface tension, the propagation speed is infinite, and such a result is unlikely to hold.

2 Reduction to the Torus

The symmetrization procedure we follow is due to Boussinesq (see [7], p.37) and has been used by Alazard, Burq and Zuily in [2] for the pure gravity case. This procedure can be describe as follows.

Take a smooth function u in \overline{M} , where we take $M = (0, l) \times \mathbf{R}$ to simplify. We can symmetrize this function with respect to the line $x_1 = 0$, giving us a function on $(-l, l) \times \mathbf{R}$, with the same values at $x_1 = -l$ and $x_1 = l$. Thus this function can be thought as periodic in x_1 , and thus as an even function on $\mathbf{T}_{2l}^1 \times \mathbf{R}$, where $\mathbf{T}_{2l}^1 := \mathbf{R}/2l\mathbf{Z}$ is the flat torus of period $2l$. For the case of $M = (0, l) \times (0, L)$, one performs successive symmetrizations along the line $x_1 = 0$ and $x_2 = 0$ (the order of these operations does not matter, obviously.) Then one can lift the function to the flat inhomogeneous Torus $\mathbf{T}_{2l, 2L}^2 := (\mathbf{R}/2l\mathbf{Z}) \times (\mathbf{R}/2L\mathbf{Z})$.

Now the problem with this symmetrization procedure is that even if the original function u is smooth, its reflexion with respect to $x_1 = 0$ is not guaranteed to be smooth. In fact, taking the example of a linear, non-constant function shows that a Lipschitz singularity appears in general. However, in the case of functions satisfying $\partial_{x_1} u = 0$, the singularity is of higher order. The exact mapping properties of this reflexion, in terms of Sobolev spaces, are summarized in the following one-dimensional Proposition from [2].

For a smooth compactly supported function v on $[0, +\infty)$, define its extension v^{ev} to \mathbf{R} by

$$(2.1) \quad v^{\text{ev}}(y) := \begin{cases} v(y) & \text{if } y \geq 0 \\ v(-y) & \text{if } y < 0. \end{cases}$$

This map obviously lift to distributions, and thus to Sobolev spaces.

Proposition 2.1. (Proposition 6.5 of [2].)

1. Assume that $0 \leq s < \frac{3}{2}$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$.
2. Assume that $\frac{3}{2} \leq s < \frac{7}{2}$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from the space $\{v \in H^s(0, +\infty), v'(0) = 0\}$ to $H^s(\mathbf{R})$.

Then we can define the symmetrized-periodized extension of a function $u \in C_0^\infty(\overline{M})$ as follows.

For $M = (0, l) \times \mathbf{R}$,

$$(2.2) \quad u^\sharp(x_1, x_2) := \begin{cases} u(x_1, x_2) & \text{if } 0 \leq x_1 \leq l \\ u(-x_1, x_2) & \text{if } -l \leq x_1 < 0 \\ u(x_1 - 2kl, x_2) & \text{if } -l + 2kl \leq x_1 \leq l + 2kl, k \in \mathbf{Z}. \end{cases}$$

This again lift to Sobolev spaces, and since the periodization procedure preserves regularity we have the following Corollary.

Corollary 2.2. Let $M = (0, l) \times \mathbf{R}$.

1. Assume that $0 \leq s < \frac{3}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H^s(M)$ to $H^s(\mathbf{T}_{2l}^1 \times \mathbf{R})$.
2. Assume that $\frac{3}{2} \leq s < \frac{7}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H_p^s(M)$ to $H^s(\mathbf{T}_{2l}^1 \times \mathbf{R})$.

Thus the regularity $s \in (3 - 3/10, 3)$ of Theorem 1.1 is sufficient for both $\eta_0 \in H_c^{s+\frac{1}{2}}(M)$ and $\psi_0 \in H_c^s(M)$ to keep their regularity by this procedure. This explains the higher limit $s < 3$.

In the case $M = (0, l) \times (0, L)$ we pose

$$(2.3) \quad u^\sharp(x_1, x_2) := \begin{cases} u(x_1, x_2) & \text{if } 0 \leq x_1 \leq l, 0 \leq x_2 \leq L \\ u(-x_1, x_2) & \text{if } -l \leq x_1 < 0, 0 \leq x_2 \leq L \\ u(x_1, -x_2) & \text{if } 0 \leq x_1 \leq l, -L \leq x_2 < 0 \\ u(-x_1, -x_2) & \text{if } -l \leq x_1 < 0, -L \leq x_2 < 0 \\ u(x_1 - 2kl, x_2 - 2KL) & \text{if } -l + 2kl \leq x_1 \leq l + 2kl, \\ & -L + 2KL \leq x_2 \leq L + 2KL, k, K \in \mathbf{Z}. \end{cases}$$

We again have the following Corollary.

Corollary 2.3. Let $M = (0, l) \times (0, L)$.

1. Assume that $0 \leq s < \frac{3}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H^s(M)$ to $H^s(\mathbf{T}_{2l, 2L}^2)$.
2. Assume that $\frac{3}{2} \leq s < \frac{7}{2}$. Then the map $u \mapsto u^\sharp$ is continuous from $H_p^s(M)$ to $H^s(\mathbf{T}_{2l, 2L}^2)$.

Now, we have reduced our initial data to Sobolev functions on the flat Torus (or the flat cylinder). In Section 3, we will show how to solve the Cauchy problem for those initial data. Suppose we have done so, obtaining functions

$$(\tilde{\eta}, \tilde{\psi}) \in C\left([0, T]; H^{s+\frac{1}{2}}(N) \times H^s(N)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(N) \times C^r(N)\right)$$

Here and in what follows,

$$(2.4) \quad \begin{aligned} N &:= \mathbf{T}_{2l}^1 \times \mathbf{R} & \text{if } M = (0, l) \times \mathbf{R}, \\ N &:= \mathbf{T}_{2l, 2L}^2 & \text{if } M = (0, l) \times (0, L). \end{aligned}$$

We now have to show that the restriction of those functions to the original base M solves the original Cauchy Problem. We start by showing that it is in the good functional space.

Lemma 2.4. *Take (η, ψ) as above, the restriction to M of the unique solution to the Cauchy problem with initial data $(\eta_0^\sharp, \psi_0^\sharp)$. Then*

$$(\eta, \psi) \in C\left([0, T]; H_p^{s+\frac{1}{2}}(M) \times H^s(M)_p\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right),$$

and $(\tilde{\eta}(t), \tilde{\psi}(t)) = (\eta^\sharp(t), \psi^\sharp(t))$ for all $t \in [0, T]$.

Proof. The fact that

$$(\eta, \psi) \in C\left([0, T]; H^{s+\frac{1}{2}}(M) \times H^s(M)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(\overline{M}) \times C^r(\overline{M})\right)$$

is evident, but we still need to prove the right angle conditions.

Take the case $M = (0, l) \times \mathbf{R}$. The water waves equation (1.3) is left invariant by the translation

$$x_1 \mapsto x_1 + 2kl, \quad l \in \mathbf{Z},$$

and by the reflection

$$x_1 \mapsto -x_1.$$

Thus, since the initial data $(\eta_0^\sharp, \psi_0^\sharp)$ is also invariant by those transformations, the solution keeps this invariance property, as a consequence of uniqueness. Now this obviously entails this right angle result.

The case $M = (0, l) \times (0, L)$ follows along the same lines. □

Now the above argument obviously applies also to the harmonic potential ϕ in the periodized fluid domain, so that $\partial_{x_1}\phi(0, x_2, y) = 0$ and the same on the other boundaries, and $\tilde{\phi} = \sharp\phi$. As a consequence, at a point $(t, x) \in [0, T] \times M$,

$$G(\eta)\psi(t, x) = G(\tilde{\eta})\tilde{\psi}(t, x).$$

Thus the functions (η, ψ) solve the original Cauchy problem, uniqueness being insured by the requirement that $(\eta^\sharp, \psi^\sharp) = (\tilde{\eta}, \tilde{\psi})$, for which uniqueness is already known. Thus the proof of Theorem 1.1 is reduced to solving the Cauchy Problem in the periodized space N .

3 The Cauchy problem in the periodized space

In the new periodized setting, the bottom is parametrized by the function b^\sharp which, since it does not satisfy any right angle condition, has only limited regularity. It however stays continuous, and this will be all that we need, taking into account (B), which becomes

$$(B') \quad \exists h_t > 0, b(x) < \eta(t, x) - h_t, \forall x \in N.$$

The following Proposition allows one to solve the water waves in N .

Proposition 3.1. *Take $N = \mathbf{T}_{2l}^1 \times \mathbf{R}$ or $N = \mathbf{T}_{2l, 2L}^2$. Suppose $s > 3 - \frac{3}{10}$ and $2 < r < s + \frac{3}{10} - 1$. Then if*

$$\left(\tilde{\eta}_0, \tilde{\psi}_0 \right) \in H^{s+\frac{1}{2}}(N) \times H^s(N)$$

satisfy condition B', there exists a time $T > 0$ depending only on the norms of those initial data and the h_0 in Assumption (B') such that the Cauchy Problem of (1.3) admits a unique solution

$$(\tilde{\eta}, \tilde{\psi}) \in C\left([0, T]; H^{s+\frac{1}{2}}(N) \times H^s(N)\right) \cap L^2\left([0, T]; C^{r+\frac{1}{2}}(N) \times C^r(N)\right),$$

satisfying Assumption (B') for each time $t \in [0, T]$.

The corresponding result was proved in ?? and ?? by Nguyen and the author, in the case of the whole space in arbitrary dimension. Most of the proof is a straightforward adaptation of those results, as we explain below. The only apparent difficulty is that the compactness properties of N would seem to preclude the use of dispersion, and thus of Strichartz estimates for such an equation with infinite speed of propagation. However, as explained below, the fact that our parametrization is constructed on semiclassical times is sufficient to recover the estimates, as in the strategy of Burq, Gerard and Tzvetkov for Schrödinger equation on compact manifolds ([8]).

The main tool used in the proof is Bony's paradifferential calculus, which is well-known to work indistinctly in euclidean space using Littlewood-Paley decomposition, or in the Torus using Fourier series. We thus can easily extend it to work on N in both cases. The first step of the proof is a paradifferential reduction of the equations, which rest on an analysis of elliptic regularity of the Dirichlet-Neumann operator, which stays identical in the periodic setting, and on paraproduct estimates. Thus, dropping the tildes in the variables for convenience, the analysis in [?] shows that the equation is equivalent to the paradifferential equation

$$(3.1) \quad \partial_t u + T_V \cdot \nabla u + iT_\gamma u = f$$

for the complex-valued unknown $u := T_p \eta + iT_q(\psi - T_B \eta)$. Here,

$$B := \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \quad V := \nabla \psi - B \nabla \eta$$

are the vertical and horizontal trace of the velocity at the surface and are thus controlled by the unknowns η and ψ , the symbols p and q are elliptic, respectively of order $1/2$ and 0 , and

smooth functions of $\nabla\eta$. The symbol γ is again elliptic, of order $3/2$, and a smooth function of $\nabla\eta$. The remainder f satisfies

$$\|f\|_{H^s} \leq \mathcal{F} \left(\|\eta\|_{H^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \left[1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|\psi\|_{W^{r,\infty}} \right].$$

Now denoting by $M_s(T)$ the supremum in time of the Sobolev norms of the solution, and by $Z_r(T)$ the L^2 norm in time of their Hölder norms, we can use classical energy estimates to prove that

$$M_s(T) \leq \mathcal{F} \left(M_s(0) + T^{\frac{1}{2}} \mathcal{F} (M_s(T) + Z_r(T)) \right).$$

Then, if we can prove that $Z_r(T)$ is bounded by the same quantity, proving contraction estimates in a lower order norm and using a quasilinear scheme of convergence as in [?] will complete the proof.

We thus only have to prove that

$$Z_r(T) \leq \mathcal{F} \left(M_s(0) + T^{\frac{1}{2}} \mathcal{F} (M_s(T) + Z_r(T)) \right),$$

which are the so-called Strichartz estimates (with loss).

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