A PARADIFFERENTIAL APPROACH FOR HYPERBOLIC DYNAMICAL SYSTEMS AND APPLICATIONS

COLIN GUILLARMOU AND THIBAULT DE POYFERRÉ,
WITH AN APPENDIX BY YANNICK GUEDES BONTHONNEAU

Abstract. We develop a paradifferential approach for studying non-smooth hyperbolic dynamics and related non-linear PDE from a microlocal point of view. As an application, we describe the microlocal regularity, i.e the H^s wave-front set for all s, of the unstable bundle E_u for an Anosov flow. We also recover rigidity results of Hurder-Katok and Hasselblatt in the Sobolev class rather than Hölder: there is $s_0 > 0$ such that if E_u has H^s regularity for $s > s_0$ then it is smooth (with $s_0 = 2$ for volume preserving 3-dimensional Anosov flows). It is also shown in the Appendix that it can be applied to deal with non-smooth flows and potentials. This work could serve as a toolbox for other applications.

1. Introduction

Consider a smooth compact manifold M (equipped with a fixed metric g) and let X be a smooth non-vanishing vector field generating an Anosov flow φ_t. This means that the tangent bundle TM has a $d\varphi_t$ invariant decomposition

$$TM = \mathbb{R}X \oplus E_u \oplus E_s$$

such that there is $C > 0$ and $\nu > 0$ so that for each $t \geq 0$

$$\begin{aligned} &\forall x \in M, \forall w \in E_u(x), \quad \|d\varphi_{-t}(x).w\| \leq Ce^{-\nu t}\|w\| \\ &\forall x \in M, \forall w \in E_s(x), \quad \|d\varphi_t(x).w\| \leq Ce^{-\nu t}\|w\|. \end{aligned} \tag{1.1}$$

The stable and unstable bundles E_s and E_u are only Hölder continuous [HPS70]. A sharp regularity statement on the Hölder exponent of E_u/E_s has been obtained by Hasselblatt [Has94, Has97] in terms of bunching of the expansions/contraction exponents of $d\varphi_t$ on E_u/E_s; we refer to these articles for a history and references of the analysis of the regularity.

A particularly interesting case is when X is the generator of the geodesic flow on a negatively curved manifold. That class fits into the more general class of *contact Anosov flows*, defined by the requirement that the Anosov form α, defined by

$$\alpha(X) = 1, \quad \alpha(E_u \oplus E_s) = 0$$

is a contact form (which means that it is smooth and $d\alpha|_{\ker\alpha}$ is non-degenerate and $i_X d\alpha = 0$). A striking rigidity result of Hurder-Katok [HK90] is that for contact Anosov flow in dimension 3 (for example the geodesic flow of a negatively curved surface), the stable and unstable bundles E_s and E_u are always[1] $C^{1,O(\mathrm{Zyg})}$, and if they are $C^{1,o(\mathrm{Zyg})}$ they are necessarily C^∞. In particular C^2 regularity implies C^∞ regularity for E_u and E_s. On the other hand Ghys [Ghy87] proved that if E_u and E_s are C^∞, then the flow is conjugate to the geodesic flow on the circle bundle of a hyperbolic surface, modulo finite covering and quotients, and smooth time change. To conclude, $E_s \in C^2$ and $E_u \in C^2$ implies that the flow is essentially a homogeneous dynamical system. The corresponding rigidity result has been proved[2] by Benoist-Foulon-Labourie [BFL92] in higher dimension provided $E_s, E_u \in C^\infty$. Hasselblatt [Has92] also proved ridigity results in any dimension, showing that if $E_u \in C^k$ for k larger than a constant depending on the bunching of the contraction/expansion exponents appearing in (1.1), then the bundles are smooth. It is a folklore conjecture that C^2-regularity of E_u, E_s for a contact Anosov flow should imply C^∞ regularity and then, using [BFL92], that the flow is smoothly conjugate to a homogeneous dynamical system.

In this work, one of the aspects we propose to study is the microlocal regularity of E_u (and E_s). By microlocal regularity we mean its regularity in phase space, i.e. T^*M, which can be encoded in the notion of H^s-wavefront set introduced by Hörmander [Hör03, Chapter 8]. Roughly speaking, it encodes where a function is singular (the position x) but also in which direction it is singular (the momentum or Fourier variable ξ decay). For example, the regularity of the unstable bundle for a Riemannian negatively curved surface Σ is encoded by a function $r \in C^\gamma(S\Sigma)$ satisfying a Ricatti equation

$$Xr + r^2 + K = 0$$

where K is the Gauss curvature. It is non-linear, and thus not very suitable for microlocal study using the standard pseudo-differential operator calculus. We thus employ the paradifferential calculus of Bony [Bon81] to approach this problem, and in particular we show that the formalism of radial estimates used recently by Dyatlov-Zworski [DZ16] for hyperbolic dynamics fits well in that picture. They allow to describe the sharp Sobolev regularity of the unstable bundle, its wave-front set and to recover the rigidity property mentionned above but in a Sobolev class: if the bundle E_s is H^s for s large enough (depending on the maximal/minimal expansions rates of the flow), then E_u is smooth provided the flow is smooth.

More precisely this method allows to show the following result. Let $E_u^*, E_s^* \subset TM$ be defined by $E_u^*(E_u \oplus \mathbb{R}X) = 0$ and $E_s^*(E_s \oplus \mathbb{R}X) = 0$. We shall say that a subbundle $E \subset TM$ is $H^s(M)$ if there are local sections $e_j : M \to E$ so that $E_j \in H^s(M)$; similarly we say that E has wavefront set in E_u^* if $\mathrm{WF}(e_j) \subset E_u^*$. We shall denote $\nu_{u/s}^{\min/\max} > 0$ the best constants so that

$$\begin{aligned} &\forall t \geq 0, \quad C^{-1} e^{-\nu_u^{\max} t} \leq \|d\varphi_{-t}|_{E_u}\| \leq C e^{-\nu_u^{\min} t}, \\ &\forall t \geq 0, \quad C^{-1} e^{-\nu_s^{\max} t} \leq \|d\varphi_{t}|_{E_s}\| \leq C e^{-\nu_s^{\min} t}. \end{aligned}$$

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[1] Here, Zyg is the Zygmund modulus of continuity.

[2] It is claimed in the paper that a C^k regularity for k large enough depending only on the dimension is sufficient.

Theorem 1. *Let X be a smooth vector field generating an Anosov flow φ_t on a closed manifold M. Then the following hold:*
1) $\mathrm{WF}(E_u) \subset E_u^*$, $\mathrm{WF}(E_s) \subset E_s^*$.
2) If X preserves a smooth measure and $\dim M = 3$. If $E_u \in H^{2+\delta}(M)$ for some $\delta > 0$, then $E_u \in C^\infty$, the same holds for E_s.
3) More generally, if $\dim M = 3$, let $s > 0$ such that there is $T > 0$ large so that uniformly on M

$$(s - \frac{3}{2})|\log \|d\varphi_T|_{E_s}\|| > \frac{1}{2} \log \|d\varphi_T|_{E_u}\|.$$

If $E_u \in H^s(M)$, then $E_u \in C^\infty(M)$.
4) In dimension $\dim M > 3$ and when X preserves a smooth measure, if $E_u \in H^s$ for $s > (\nu_u^{\max} + \nu_s^{\max})/\nu_s^{\min}$, then $E_u \in C^\infty$.

If the vector field is C^α for $\alpha > 1$, the same results hold by replacing WF *by the H^α wavefront set WF_{H^α} in 1), $E_u \in C^\infty$ by $E_u \in H^{\alpha-\delta}(M)$ for all $\delta > 0$ in 2), 3) and 4).*

This result improves slightly the result of Hasselblatt [Has92] in the sense that we obtain a statement on the wave-front set of the bundle and the rigidity result holds in Sobolev spaces rather than Hölder spaces (recall that $H^{2+\delta}$ is only included in $C^{3/2}$ in $\dim M = 3$).

The method we employ is based on propagation estimates: we use Bony's propagation of singularity for paradifferential operators [Bon81] and we show a version of the radial point estimates for paradifferential operators. This method gives a sharp regularity statement for the bundle in the Sobolev classes, but this is not very interesting as it is weaker than the results of Hasselblatt [Has94] which hold in Hölder norms. Radial point estimates in the classical cases were introduced by Melrose [Mel94] and developed also by Vasy [Vas13], Dyatlov-Zworski [DZ16, DZ19] in the particular settings of Anosov flows, see also Dyatlov-Guillarmou [DG16] for more general cases (Axiom A flows), Wang [Wan] in Besov spaces and Guedes Bonthonneau-Lefeuvre [GBL] in the Hölder Zygmund classes.

Our method also allows to understand the sharp Sobolev regularity and rigidity phenomena for solutions U of a general Riccati equation of the form

$$\mathcal{L}_X U + Q(x, U) = 0$$

where U is a Hölder sections of an $\mathrm{End}(E)$ for some smooth bundle E on which the flow φ_t admits a (linear) lifted action $\tilde{\varphi}_t$, Q is a quadratic polynomial in U with smooth coefficients in x, $\mathcal{L}_X$ is the Lie derivative in the direction of an Anosov vector field X and $f \in H^s(M)$ for some $s > 0$. If the regularity of U is larger than a certain threshold s depending on $\nu_{s/u}^{\min/\max}$ and on $\partial_U Q(\cdot, U)$, then it is smooth. More generally, the method applied to general nonlinearities, i.e. equations of the form $\mathcal{L}_X U + F(x, U) = 0$ for some smooth functional $F : M \times E \to \mathbb{R}$. See Remark 4.

In turn, paradifferential methods are well-suited to study PDEs with non-smooth coefficients, and we also show in Section 4 for Holder potential (Proposition 4.1), and more generally in the Appendix by Guedes Bonthonneau for Hölder flows and potentials (Theorem

4), that this provides a good microlocal method for analyzing the resolvent $(X+V+\lambda)^{-1}$ of non-smooth Anosov vector field X and non-smooth potentials V. In particular, combining the paradifferential calculus with the microlocal approach of Faure-Sjöstrand [FS11], one recovers the fact that a theory of Ruelle resonances can be done also for Hölder flows and potentials in a spectral strip of finite size, as was shown by Butterley-Liverani [BL07] for Anosov flows, or by Blank-Keller-Liverani [BKL02], Baladi-Tsujii [BT07] and Gouëzel-Liverani [GL08] for hyperbolic diffeomorphisms. The case of non-smooth potential is quite natural as the geometric potentials given by the unstable or stable Jacobians are only Hölder, the proof for a smooth flow is explained in Section 4 and the general case for a Hölder Anosov flow is explained by Guedes Bonthonneau in the Appendix using the construction of escape function for non-smooth flows in [GB18].

We hope that more generally that this approach with paradifferential calculus could be applied more systematically for non-smooth dynamics and related non-linear equations.

Acknowledgements. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 725967). This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while C.Guillarmou was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2019 semester. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the T. De Poyferré participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the spring 2021 semester.

2. Paradifferential and $S_{1,1}$ Calculi

Paradifferential calculus is a way of blending nonlinear harmonic analysis with microlocal techniques. From the pseudodifferential viewpoint, it corresponds to quantization of symbols belonging to Hörmander's forbidden class $S_{1,1}$, but satisfying some conditions salvaging the existence of a calculus. We start with a review of this calculus on manifolds, since such a reference does not seem to be available. The classical references for the Euclidean case are the original papers of Bony [Bon81], Meyer [Mey81], Bourdaud [Bou88], and Hörmander [Hör88, Hör97]. We also refer to the book of Taylor [Tay91] that brings a discussion on manifold. In what follows, M will be a compact, smooth manifold of dimension n, and we equip it with a smooth Riemannian metric g_0 that allows to put norms on the fibers of the tangent and cotangent bundles TM and T^*M, as well as a smooth Riemannian measure dv_g on M. The Sobolev spaces $H^s(M)$ can then be defined as $H^s(M) := \{u \in C^{-\infty}(M) \,|\, (1+\Delta_{g_0})^{s/2} u \in L^2(M)\}$, or equivalently $u \in H^s(M)$ if and only if $u|_{U_j} \in H^s(U_j)$ for all j where $(U_j)_j$ is a covering by charts diffeomorphic to the unit ball $B(0,1) \subset \mathbb{R}^n$. The Hölder space $C^\alpha(M)$ is simply $C^k(M)$ is $\alpha = k \in \mathrm{N}$, and consists more generally of functions f whose $k := [\alpha]$ derivative is $(\alpha - k)$-Hölder in charts, i.e $\sup_{x\neq y} |f(x)-f(y)|/|x-y|^\alpha < \infty$.

2.1. Hörmander $\tilde{\Psi}_{1,1}$ Calculus.

Definition 2.1. For an order $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, the class $S^m_{\rho,\delta}(M)$ consists in the space of smooth functions $a(x,\xi)$ on the cotangent bundle T^*M, satisfying estimates of the form

$$\forall \alpha, \beta, \exists C_{\alpha,\beta} > 0, \forall (x,\xi) \in T^*M, \quad \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|} < \infty,$$

in local charts on M, and for any multi-indices α and β.

The set $S_{1,1}(M) = \cup_{m\in\mathbb{R}} S^m_{1,1}(M)$ is then a filtered Fréchet $*$-algebra, with seminorms given by the best $C_{\alpha,\beta}$ in the estimates above. It is invariant by changes of coordinates.

It contains the symbols of differential operators. The residual symbols $a \in S^{-\infty}(M) = \cap_{m\in\mathbb{R}} S^m_{1,1}(M)$ are those smooth functions which are decaying faster than any $\langle\xi\rangle^{-N}$ for $N > 0$ in the fibers of T^*M.

One can now define a quantization Op on M by using a partition of unity $(\psi_j)_j$ associated to a finite family of local charts $U_j \simeq B(0,\varepsilon) \subset \mathbb{R}^n$, some functions $\tilde{\psi}_j \in C_c^\infty(U_j)$ so that $\tilde{\psi}_j = 1$ on $\mathrm{supp}(\psi_j)$, and using the left quantization in charts

$$\mathrm{Op}(a)f(x) = \sum_j \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{U_j} e^{i(x-y).\xi} \tilde{\psi}_j(x) a(x,\xi) \psi_j(y) f(y) dy d\xi.$$

We define the classes $\Psi^m_{1,1}(M)$ to be the set of operators A mapping $C^\infty(M)$ to $C^{-\infty}(M)$ continuously so that $A = \mathrm{Op}(a) + S$ for some $a \in S^m_{1,1}(M)$ and S a smoothing operator, i.e. an operator with smooth Schwartz kernel. The resulting operators are invariant by change of variables, as we shall see below. The class $\Psi^m_{1,1}(M)$ is not very convenient since there is no good composition law, it is not bounded on $H^s(M)$ for $s \leq 0$ and it is not stable when taking adjoints. However, there is a notion of principal symbol. To see that, recall that the formula for action of coordinate changes on the symbol of a pseudodifferential operators in $\mathbb{R}^n$ does not involve spatial differentiation of the symbol:

Lemma 2.2. *For $a \in S^m_{1,1}(\mathbb{R}^n)$ compactly supported in x, $\chi \in C_c^\infty(\mathbb{R}^n)$ and ϕ a smooth diffeomorphism of $\mathbb{R}^n$ equal to the identity outside a compact set,*

$$(\phi^{-1})^* a(x, D_x) \chi(x) \phi^* = a_\phi(x, D_x) \chi \circ \phi^{-1},$$

where $a_\phi \in S^m_{1,1}(\mathbb{R}^n)$ is such that

$$a_\phi(\phi(x), (d\phi(x)^{-1})^T \xi) - a(x,\xi) \in S^{m-1}_{1,1}(\mathbb{R}^n).$$

Proof. We can use the formula for the change of variable for a left-quantized symbol given by [Shu01, Theorem 4.1]: if $A_1 = (\phi^{-1})^* a(x, D_x) \chi(x) \phi^*$

$$A_1 u(x) = \int e^{i(x-y)\xi} a(\phi^{-1}(x), \psi(x,y)\xi) \chi(\phi^{-1}(y)) |\det(\psi(x,y))| \det d\phi^{-1}(y)| u(y) dy d\xi$$

for some smooth map $\psi(x,y) \in \mathrm{GL}_n(\mathbb{R})$ so that $(\phi^{-1}(x) - \phi^{-1}(y)).\psi(x,y)\xi = (x-y).\xi$. As in the proof of [Shu01, Theorem 3.1], this can be rewritten under the form $A_1 u(x) = \int_{\mathbb{R}^n} e^{i(x-y).\xi} a_\phi(x,\xi) \chi(\phi^{-1}(y)) u(y) dy d\xi$ where

$$a_\phi(x,\xi) = \int\int e^{i(x-y).(\eta-\xi)} a(\phi^{-1}(x), \psi(x,y)\eta) |\det(\psi(x,y))| \det d\phi^{-1}(y)| d\eta dy$$

and [Shu01, Theorem 4.1][3] shows that this is in $S^m_{1,1}(\mathbb{R}^n)$ with an expansion as $|\eta|\to\infty$

$$a_\phi(\phi(x),\eta)\sim\Big(\sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a(x,d\phi(x)^T\eta)\cdot D_z^\alpha e^{i\phi''_x(z).\eta}|_{z=x}\Big)\chi(\phi^{-1}(x))$$

with $\phi''_x(z)=\phi(z)-\phi(x)-d\phi(x)(z-x)$. The asymptotic expansion makes sense since each ξ derivative improves the η decay by one order and $\phi''_x(z)$ vanishes to second order at $z=x$. $\square$

As a direct consequence of this formula, we can define a principal symbol map:

Definition 2.3. On a compact manifold M, the principal symbol map

$$\sigma:\Psi^m_{1,1}(M)\to S^m_{1,1}(M)/S^{m-1}_{1,1}(M)$$

is well defined, with kernel $\Psi^{m-1}_{1,1}(M)$.

However, without additional restrictions on the symbols, this map is useless as it is not an algebra homomorphism.

Hörmander [Hör88] introduced a subclass $\tilde\Psi_{1,1}(\mathbb{R}^n)$ of $\Psi_{1,1}(\mathbb{R}^n)$ that is stable by composition and adjoint, and is bounded on $H^s(\mathbb{R}^n)$ for all $s\in\mathbb{R}$. Let us recall its definition and properties, and exlain ho this extends to a compact manifold.

Let $\chi\in C^\infty(\mathbb{R}^{2n})$ satisfying $\chi(\eta,\xi)=1$ in $\{|\eta|\le|\xi|/2,|\xi|\ge 2\}$ and $\operatorname{supp}\chi\subset\{|\eta|<|\xi|,|\xi|\ge 1\}$. We say that $a\in\tilde S^m_{1,1}(\mathbb{R}^n)$ if for all $\varepsilon>0$ small, the function

$$a_\varepsilon(x,\xi):=\frac{1}{(2\pi)^n}\int\int e^{i(x-y).\eta}\chi(\xi+\eta,\varepsilon\xi)a(y,\xi)dyd\eta=\mathcal{F}^{-1}_{\eta\to x}(\chi(\xi+\eta,\varepsilon\xi)\mathcal{F}(a(\cdot,\xi))(\eta))\tag{2.1}$$

satisfies the bounds for all α,β,N

$$|\partial_x^\alpha\partial_\xi^\beta a_\varepsilon(x,\xi)|\le C_{\alpha\beta N}\varepsilon^N(1+|\xi|)^{m+|\alpha|-|\beta|}.$$

In other words, the microlocalisation of $a(x,\xi)$ (in x) in small cones near the twisted diagonal $\xi+\eta=0$ enjoys some decay to all order in the topology of $S^m_{1,1}(\mathbb{R}^n)$. We define $\tilde\Psi^m_{1,1}(\mathbb{R}^n):=\operatorname{Op}(\tilde S^m_{1,1}(\mathbb{R}^n))$ where Op is the usual left quantization on $\mathbb{R}^n$.

Proposition 2.4. [Hör88, Theorem 3.6 and Theorem 4.2], [Bou88, Theorème 3] *An operator $A\in\Psi^m_{1,1}(\mathbb{R}^n)$ belongs to $\tilde\Psi^m_{1,1}(\mathbb{R}^n)$ if and only if $A^*\in\Psi^m_{1,1}(\mathbb{R}^n)$. An operator $A\in\Psi^m_{1,1}(\mathbb{R}^n)$ belongs to $\tilde\Psi^m_{1,1}(\mathbb{R}^n)$ if and only if A is bounded as a map $H^{s+m}(\mathbb{R}^n)\to H^s(\mathbb{R}^n)$ for all $s\in\mathbb{R}$.*

For this Proposition, Bourdaud [Bou88] proved that the largest subalgebra of those operators $A\in\mathcal{L}(L^2)$ contained $\Psi^0_{1,1}(\mathbb{R}^n)$ is the space of those $A\in\Psi^0_{1,1}(\mathbb{R}^n)$ so that $A^*\in\Psi^0_{1,1}(\mathbb{R}^n)$, and Hörmander [Hör88] gave the $\tilde\Psi_{1,1}(\mathbb{R}^n)$ characterization and the necessary conditions for boundedness on $H^s(\mathbb{R}^n)$ for all $s\in\mathbb{R}$.

[3]Shubin does not technically considers the case of $S^m_{1,1}(\mathbb{R}^n)$ but the argument readily applies to that case as well.

Since for a smooth diffeomorphism ϕ equal to Id outside a compact set, we have $\phi^* : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, we deduce from Proposition 2.4 that for $a \in \tilde{S}^m_{1,1}(\mathbb{R}^n)$ compactly supported in x, and ϕ a diffeomorphism of $\mathbb{R}^n$ equal to the identity outside a compact set,

$$(\phi^{-1})^* a(x, D_x)\phi^* \in \tilde{\Psi}^m_{1,1}(\mathbb{R}^n).$$

This implies that one can define the same class on a compact manifold M:

Definition 2.5. We say that $A \in \tilde{\Psi}^m_{1,1}(M)$ if $A \in \Psi^m_{1,1}(M)$ and for each j, $\tilde{\psi}_j A \psi_j \in \tilde{\Psi}^m_{1,1}(\mathbb{R}^n)$, i.e. its symbol in the charts U_j is the restriction elements in $\tilde{S}^m_{1,1}(\mathbb{R}^n)$ to U_j.

Due to the invariance of $\tilde{\Psi}^m_{1,1}(\mathbb{R}^n)$ under diffeomorphism, we see that this definition is independent of the choice of charts.

The wave-front set $\mathrm{WF}(A)$ of such an operator is then defined as in the classical case as the conic set of points (x_0, ξ_0) which do not admit a conic neighborhood in T^*M where the symbol $a(x, \xi)$ in charts is an $\mathcal{O}_{C^N}(\langle \xi \rangle^{-N})$ for all $N > 0$; its properties are identical as for the classical class $\Psi_{1,0}(M)$.

It has been known since Stein that $\mathrm{Op}(S^m_{1,1}(\mathbb{R}^n))$ maps $H^{s+m}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ for any $s > 0$, but there are counter-examples in the case $s = 0$. However, the class $\tilde{\Psi}_{1,1}(M)$ has all the desired properties. From the boundedness and composition properties on $\mathbb{R}^n$, we directly obtain the following results on M:

Proposition 2.6. *The following properties hold:*

- [Bou88, Theorem 1]*, following Stein's proof. Let $A \in \Psi^m_{1,1}(M)$, then $A : H^{s+m}(M) \to H^s(M)$ is bounded for all $s > 0$ and $A : C^{s+m}(M) \to C^s(M)$ if $s, s+m > 0$ are non-integer.*
- [Hör88, Theorem 3.6] *If $A \in \tilde{\Psi}^m_{1,1}(M)$, then A maps $H^{s+m}(M)$ to $H^s(M)$ for any $s \in \mathbb{R}$.*
- [Hör88, Theorem 5.2] *If $A \in \Psi^\mu_{1,1}(M)$, $B \in \tilde{\Psi}^m_{1,1}(M)$ and $C \in \tilde{\Psi}^\mu_{1,1}(M)$ then $AB \in \Psi^{m+\mu}_{1,1}(M)$ and $BC \in \tilde{\Psi}^{m+\mu}_{1,1}(M)$.*
- [Hör88, Theorem 4.2] *If $A \in \tilde{\Psi}^m_{1,1}(M)$ then $A^* \in \tilde{\Psi}^m_{1,1}(M)$.*

In fact, operators in $\tilde{\Psi}^0_{1,1}(M)$ are Calderón-Zygmund and thus act on $L^p(M)$ for $1 < p < \infty$, Besov, Hardy and BMO spaces; see the book [Tay91] for very general results in this direction.

Although $\tilde{\Psi}_{1,1}(M)$ is an algebra, it is still too big for a calculus to exist, in the sense that the principal symbol is not a homomorphism. Thus we need to introduce smaller subspaces of symbols:

Definition 2.7. For $r > 0$, and $m \in \mathbb{R}$ the space of r-regular symbols ${}^rS^m_{1,1}(M)$, is the space of $a \in S^m_{1,1}(M)$, such that in local charts

$$\forall \alpha, \beta, \exists C_{\alpha,\beta} > 0, \forall (x, \xi) \in T^*M, \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \le C_{\alpha,\beta} \langle \xi \rangle^{m + (|\alpha| - r)_+ - |\beta|},$$

with $f_+ = \max(0, f)$. We also define ${}^r\tilde{S}^m_{1,1}(M) := {}^rS^m_{1,1}(M) \cap \tilde{S}^m_{1,1}(M)$.

Note that for $r \geq r' \geq 0$, we have ${}^{r-r'}S^{m-r'}_{1,1}(M) \subset {}^rS^m_{1,1}(M)$ and that for $a \in {}^rS^m_{1,1}(M)$, $b \in {}^rS^{m'}_{1,1}(M)$ one has

$$ab \in {}^rS^{m+m'}_{1,1}(M), \quad \partial_\xi a \in {}^rS^{m-1}_{1,1}(M), \quad \partial_x a \in {}^{r-1}S^m_{1,1}(M). \tag{2.2}$$

If $S^m(M) = S^m_{1,0}(M)$ denotes the standard class of symbols satisfying $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$, and if $0 \leq r \leq r'$, we also have

$$S^m(M) = {}^\infty\tilde{S}^m_{1,1}(M) \subset {}^{r'}\tilde{S}^m_{1,1}(M) \subset {}^r\tilde{S}^m_{1,1}(M) \subset {}^0\tilde{S}^m_{1,1}(M) = \tilde{S}^m_{1,1}(M).$$

The residual symbols are those in ${}^\infty\tilde{S}^{-\infty}_{1,1}(M) = S^{-\infty}(M)$. We define ${}^r\Psi^m_{1,1}(M)$ (resp. ${}^r\tilde{\Psi}^m_{1,1}(M) \subset \tilde{\Psi}^m_{1,1}(M)$) to be the operators which can be written as $\mathrm{Op}(a) + S$ with $a \in {}^rS^m_{1,1}(M)$ (resp. $a \in {}^r\tilde{S}^m_{1,1}(M)$) and S a smoothing operator (with C^∞ Schwartz kernel). As in the proof of Lemma 2.2, we directly see that being in ${}^r\Psi^m_{1,1}(M)$ or in ${}^r\tilde{\Psi}^m_{1,1}(M)$ is independent of the choice of charts and coordinates. We also denote by $\Psi^m(M)$ the set of operators that are quantizations of symbols in $S^m(M)$. The proof of Lemma 2.2 also shows that the principal symbol σ is a well-defined linear map

$$\sigma : {}^r\Psi^m_{1,1}(M) \to {}^rS^m_{1,1}(M)/{}^rS^{m-1}_{1,1}(M)$$

with $\ker \sigma = {}^r\Psi^{m-1}_{1,1}(M)$.

For $r > 0$, we have a calculus for ${}^r\tilde{\Psi}^m_{1,1}(M)$, modulo operators in $\tilde{\Psi}^{m-r}_{1,1}(M)$. More precisely, the formulas for the standard quantization in $\mathbb{R}^n$ will all be truncated at order $(m-r)$ as explained now:

Proposition 2.8. [Hör88, Theorem 6.2 and Theorem 6.4] *For $r > 0$, the following properties hold:*

- *Consider $a \in S^m_{1,1}(\mathbb{R}^n)$ and $b \in {}^r\tilde{S}^{m'}_{1,1}(\mathbb{R}^n)$ with compact support in x. Then we have*
$$a(x, D_x)b(x, D_x) = c(x, D_x),$$
where the symbol $c \in S^{m+m'}_{1,1}(\mathbb{R}^n)$ satisfies
$$c(x,\xi) = \sum_{|\alpha| < \lceil r \rceil} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi) + S^{m+m'-r}_{1,1}(\mathbb{R}^n).$$
In particular, if $a \in {}^rS^m_{1,1}(\mathbb{R}^n)$, then $c \in {}^rS^{m+m'}_{1,1}(\mathbb{R}^n)$.
- *For $a \in {}^r\tilde{S}^m_{1,1}(\mathbb{R}^n)$ with compact support in x, we have*
$$a(x, D_x)^* = b(x, D_x),$$
where $b \in {}^r\tilde{S}^m_{1,1}(\mathbb{R}^n)$ satisfies
$$b(x,\xi) = \sum_{|\alpha| < \lceil r \rceil} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \bar{a}(x,\xi) + \tilde{S}^{m-r}_{1,1}(\mathbb{R}^n).$$

Proof. Using the definition of ${}^r\tilde{S}^m_{1,1}(\mathbb{R}^n)$, the property for the adjoint follows from [Hör88, Theorem 6.2] by taking $N = \lceil r \rceil$ so that $m_N = m - r$ in that Theorem. The property for the composition follows from [Hör88, Theorem 6.4] by also setting $N = \lceil r \rceil$. The fact that $c \in {}^r S^{m+m'}_{1,1}(\mathbb{R}^n)$ uses that $S^{m+m'-r}(\mathbb{R}^n) \subset {}^r S^{m+m'}_{1,1}(\mathbb{R}^n)$ and the properties (2.2). □

As a direct consequence, on M we obtain the principal calculus for $r \geq 1$, and the subprincipal one for $r \geq 2$ (recall below that $\sigma(A)$ for $A \in \Psi^m_{1,1}$ is always a class modulo $S^{m-1}_{1,1}(M)$).

Proposition 2.9. *The following hold true:*

- *If $A \in \Psi^m_{1,1}(M)$, resp. $A \in \tilde{\Psi}^m_{1,1}(M)$, and $B \in {}^r\tilde{\Psi}^{m'}_{1,1}(M)$ for $r > 0$, then $AB \in \Psi^{m+m'}_{1,1}(M)$, resp. $AB \in \tilde{\Psi}^{m+m'}_{1,1}(M)$, and*
$$\sigma(AB) = \sigma(A)\sigma(B) \mod S^{m+m'-r}_{1,1}(M)$$
If in addition $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$, then $AB \in {}^r\tilde{\Psi}^{m+m'}_{1,1}(M)$. If moreover $r \geq 1$
$$\sigma(AB) = \sigma(A)\sigma(B) \mod {}^{r-1}\tilde{S}^{m+m'-1}_{1,1}(M).$$
- *If $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$ and $B \in {}^r\tilde{\Psi}^{m'}_{1,1}(M)$ for $r \geq 1$ then $[A, B] \in {}^{r-1}\tilde{\Psi}^{m+m'-1}_{1,1}(M)$ and*
$$\sigma([A, B]) = \frac{1}{i}\{\sigma(A), \sigma(B)\} \mod \tilde{S}^{m+m'-(r-1)}_{1,1}(M),$$
where $[.,.]$ is the commutator and $\{.,.\}$ the Poisson bracket of functions on T^*M.
- *If $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$ for $r > 0$, then for any inner product on M, the adjoint $A^* \in {}^r\tilde{\Psi}^m_{1,1}(M)$ and*
$$\sigma(A^*) = \overline{\sigma(A)} \mod \tilde{S}^{m-r}_{1,1}(M).$$
If $\sigma(A)$ is real and $r \geq 1$, then $A^* - A \in {}^{r-1}\tilde{\Psi}^{m-1}_{1,1}(M)$.

A direct consequence of this is the following microlocal property: if $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$, $B, B' \in \Psi^0(M)$, then
$$\mathrm{WF}(B) \cap \mathrm{WF}(B') = \emptyset \Longrightarrow BAB' \in \tilde{\Psi}^{m-r}_{1,1}(M). \tag{2.3}$$

The construction of parametrices for elliptic operators being purely symbolic, it still works for our operators. The *elliptic set* of an operator $A \in \tilde{\Psi}^m_{1,1}(M)$ is defined just as in the classical case: $(x_0, \xi_0) \in \mathrm{ell}(A) \subset T^*M$ if there is $C > 0$ such that $|\xi|^{-m}|\sigma(A)(x, \xi)| \geq C^{-1} > 0$ for all (x, ξ) in a conic neighborhood V of (x_0, ξ_0) with $|\xi| > C$.

Proposition 2.10. *If (x_0, ξ_0) is in the elliptic set of $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$, with $r > 0$, there exists $B \in \Psi^{-m}_{1,1}(M)$, $P \in \Psi^0(M)$ elliptic near (x_0, ξ_0) and $E \in \Psi^{-r}_{1,1}(M)$, with $BA = P + E$.*

Proof. The proof is contained in [Tay91, Theorem 3.4.B] and follows directly from Proposition 2.9. We sketch the proof for the convenience of the reader. Let $\chi \in C^\infty(T^*M)$ homogeneous of degree 0 in $|\xi| > 1$, supported in a cone where a is elliptic and equal to 1 near $\{(x_0, \lambda\xi_0) \,|\, \lambda > 1\}$. Let $b_0(x, \xi) := \chi(x, \xi)/\sigma(A)(x, \xi)$ for $|\xi|$ large enough and extend b_0 in T^*M in a smooth fashion; note that $b_0 \in {}^rS^{-m}_{1,1}(M)$. We set $B_0 = \mathrm{Op}(b_0)$ and $P = \mathrm{Op}(\chi) \in \Psi^0(M)$, then by Proposition 2.9 we get $E_0 := B_0A - P \in \Psi^{-1}_{1,1}(M) \cup \Psi^{-r}_{1,1}(M)$. If $r \geq 1$, we set $b_1 = \chi\sigma(E_0)/\sigma(A)$ for $|\xi|$ large enough and $B_1 = \mathrm{Op}(b_1)$ we obtain $E_1 = B_1A - E_0 \in \Psi^{-2}_{1,1}(M) \cup \Psi^{-r}_{1,1}(M)$. We can continue inductively the parametrix and construct B_j for $j \leq r$ and set $B = \sum_{j=0}^{\lceil r\rceil} B_j$ until we reach a remainder in $\Psi^{-r}_{1,1}(M)$. $\square$

From this and Proposition 2.6, one deduces directly the associated elliptic estimates (see [Tay91, Theorem 3.4.D] for reference):

Corollary 2.11. *Let $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$ with $m \geq 0$ and $(x_0, \xi_0) \in \mathrm{ell}(A)$. Let $u \in H^{s'}(M)$ for some $s' > 0$ and assume that $Au \in H^s(M)$ for $s > s' - m$. Then for each $Q \in \Psi^0(M)$ microsupported in a small enough conic neighborhood of (x_0, ξ_0), $Qu \in H^{\min(s+m,s'+r)}(M)$ and there is $C > 0$ independent of u so that*

$$\|Qu\|_{H^{\min(s+m,s'+r)}(M)} \leq C\|Au\|_{H^s(M)} + C\|u\|_{H^{s'}(M)}.$$

Proof. By Proposition 2.10, there exists $B \in \Psi^{-m}_{1,1}(M)$, $P \in \Psi^0(M)$ elliptic near (x_0, ξ_0) and $E \in \Psi^{-r}_{1,1}(M)$ such that $BA = P + E$. Then $Pu = BAu - Eu$ with $BAu \in H^{s+m}(M)$ and $Eu \in H^{s'+r}(M)$ by Proposition 2.6 (note that $s + m > 0$). Using standard pseudo-differential calculus in $\Psi(M)$ and ellipticity of P near (x_0, ξ_0), we then obtain the desired result. $\square$

Next, we state the sharp Gårding inequality in that setting:

Proposition 2.12. [Bon81, Theorème 6.8],[Hör88, Theorem 7.1] *Let $A \in {}^r\tilde{\Psi}^m_{1,1}(M)$ with $r \in (0, 2]$ and assume that $\mathrm{Re}(\sigma(A))(x, \xi) \geq 0$ for all $x \in M$ and ξ large enough, then there is $C > 0$ such that for any $u \in C^\infty(M)$,*

$$\mathrm{Re}\,\langle Au, u\rangle \geq -C\,\|u\|^2_{H^{m/2-r/4}(M)}\,.$$

Extension to operators acting on vector bundles. As for the usual pseudo-differential operators in the class $\Psi^m(M)$, the theory extends in the obvious way on a smooth vector bundle $E \to M$ equipped with a Hermitian scalar product $\langle\cdot,\cdot\rangle_E$. The main difference is that the symbols are with values in the endomorphism bundle $\mathrm{End}(E) = E \otimes E^* \to M$. The only change is the fact that in Proposition 2.9, one has $[A, B] \in {}^{r-1}\tilde{\Psi}^{m+m'-1}_{1,1}(M)$ only if $[\sigma(A), \sigma(B)] = 0$ as elements of $\mathrm{End}(E)$: this is the case for example if A has principal symbol $a(x, \xi) \otimes \mathrm{Id}$ for some function $a \in {}^r\tilde{S}^m_{1,1}(M)$. The definition of elliptic set has also to be replaced by $(x_0, \xi_0) \in \mathrm{ell}(A)$ if and only there is conic neighborhood V of (x_0, ξ_0) such that $\sigma(A)(x, \xi)$ is invertible in $\mathrm{End}(E)$ for $|\xi|$ large enough. For the sharp Gårding inequality (Proposition 2.12), the condition $\mathrm{Re}(\sigma(A))(x, \xi) \geq 0$ has to be understood as $\sigma(A)(x, \xi) + \sigma(A)^*(x, \xi) \geq 0$ in the sense of symmetric endomorphisms on E for the scalar product $\langle\cdot,\cdot\rangle_E$; indeed, the proof of [Hör88], strongly based on the proof of the classical case [Hör07, Theorem 18.1.14] applies equally for quantization of symbols with values in Banach spaces, as mentioned in [Hör07, Remark 2, Page 79].

2.2. **Paradifferential Calculus.** The paradifferential calculus is a way of regularising operators with rough coefficients, which turns out to be well suited to study non linear expressions. Since we shall only need this case, we will consider the case of differential operators.

Following [Tay91, Section 1.3], let us first introduce the class of symbols with rough coefficients: we say that $a \in C^r S^m_{1,\delta}(M)$ with $r \geq 0$ and $\delta \in [0,1]$ if $\partial_\xi^\beta a \in C^r(T^*M)$ for all β with the following bounds in local charts

$$\forall \beta, \exists C_\beta > 0, \quad \|\partial_\xi^\beta p(\cdot,\xi)\|_{C^r} \leq C_\beta (1+|\xi|)^{m-|\beta|+r|\delta|}$$

with the usual convention that when $r \in \mathrm{N}$, the C^r norm involves the C^0 norm of the $j \leq r$ derivatives. We shall also denote $C^r S^m(M) := C^r S^m_{1,0}(M)$.

Definition 2.13. For an order $m \in \mathrm{N}$, and an index $r \geq m$, ${}^r\mathrm{Diff}^m(M)$ denotes the set of differential operators P of order m on M which in local charts can be written under the form

$$P = \sum_{|\alpha|\leq m} p_\alpha(x)\partial_x^\alpha$$

with $p_\alpha \in C^{r-(m-|\alpha|)}(M)$. They can also be written under the form $\mathrm{Op}(p)$ where $p \in C^{r-m}(T^*M)$ is polynomial of order m in the fibers, and given in the charts by $p(x,\xi) = \sum_{|\alpha|\leq m} p_\alpha(x)(i\xi)^\alpha$. Note that p belongs to $\oplus_{k=0}^m C^{r-k}S^{m-k}(M)$.

This class contains the set of differential operators of order m with C^r coefficients. We also remark that if $P \in {}^r\mathrm{Diff}^m(M)$, then $P^* \in {}^r\mathrm{Diff}^m(M)$ so that this class is stable by taking adjoint, contrary to the space of differential operators with C^r coefficients.

Lemma 2.14. *There is a continuous linear map sending each differential operator* $P = \mathrm{Op}(p) \in {}^r\mathrm{Diff}^m(M)$ *to a symbol* $p^\sharp \in {}^r\tilde{S}^m_{1,1}(M)$*, in such a way that* $p^\flat := p - p^\sharp \in C^{r-m}S^{m-r}_{1,1}(M)$ *and* $p^\flat \in \oplus_{k=0}^m \cap_{s\in(0,r-k)} C^{r-k-s}S^{m-k-s}(M)$*. Moreover, if* $m \in \{0,1\}$*,*

$$\forall s \in (0,r), \quad \mathrm{Op}(p^\flat) : H^{s+m-r}(M) \to H^s(M) \tag{2.4}$$

is bounded.

Proof. The proof is done in [Bon81, Section 2], [Tay91, Section 3.2] or [Hör97, Chapter X] in $\mathbb{R}^n$, we just have to proceed similarly using local charts, as is done in [CG20, Section 4.1][4]. We fix a covering with small charts $U_j \simeq B(0,\varepsilon) \subset \mathbb{R}^n$ and an associated partition of unity $(\psi_j)_j$. In each chart U_j, a function $a_j \in C^r_c(U_j)$ can be regularized as in [Tay91, Section 3.2]: let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ so that $\chi(\eta,\xi) = 0$ for $|\eta| > |\xi|/2$ and $\chi(\eta,\xi) = 1$ if $|\eta| < |\xi|/16$, and set for $x \in U_j, \xi \in \mathbb{R}^n$

$$a_j^\sharp(x,\xi) = \mathcal{F}^{-1}_{\eta\to x}(\hat{a}_j(\eta)\chi(\eta,\xi)).$$

Then we define $a^\sharp = \sum_j \psi_j a_j^\sharp$ where $a_j := a|_{U_j} \times \tilde{\psi}_j$ and $\tilde{\psi}_j \in C^\infty_c(U_j)$ is equal to 1 on $\mathrm{supp}(\psi_j)$. For a differential operator p, we proceed similarly by writting $P = \mathrm{Op}(p)$ and

[4]In this paper, the regularization is with respect to a semi-classical parameter rather than $|\xi|$

$p(x,\xi)|_{U_j}\tilde{\psi}_j(x) = \sum_{|\alpha|\le m} a_{j,\alpha}(x)(i\xi)^\alpha$ and setting

$$p^\sharp(x,\xi) = \sum_j \sum_{|\alpha|\le m} \psi_j(x) a^\sharp_{j,\alpha}(x,\xi)(i\xi)^\alpha. \tag{2.5}$$

According to [Tay91, Proposition 3.2.1, Proposition 1.3.B], one has, setting $k = m - |\alpha|$

$$a^\sharp_{j,\alpha} \in {}^{r-k}\tilde{S}^0_{1,1}(U_j), \quad \psi_j(a_{j,\alpha} - a^\sharp_{j,\alpha}) \in C^{r-k}S^{-r+k}_{1,1}(U_j) \cap C^{r-k-s}S^{-s}(U_j)$$

for $s \in (0, r-k)$. Thus $p^\sharp \in {}^r\tilde{S}^m_{1,1}(M)$ and $p^\flat = p - p^\sharp$ has the announced regularity. For (2.4), we refer to [Tay91, Theorem 2.1.A]. □

Remark. The regularization procedure described above is of course depending on the choice of cutoff χ but two regularizations of a symbol $a \in C^rS^m(M)$ yield the same symbol $a^\sharp \in {}^r\tilde{S}^m_{1,1}(M)$ modulo the class $\tilde{S}^{m-r}_{1,1}(M)$: this is proved in [Bon81, Theorème 2.1] or in [Hör97, Proposition 10.2.2.]. Another equivalent way of regularizing can be done using Littlewood-Paley decomposition with the same exact properties, see again [Bon81, Theorem 2.1] or [Tay91, Section 3.2]. We can thus use freely both constructions for the paradifferential operator, keeping in mind that some proof are sometime more transparent with one definition that the other.

Definition 2.15. For a differential operator $P = \mathrm{Op}(p) \in {}^r\mathrm{Diff}^m(M)$, we define its associated paradifferential operator to be

$$T_p := \mathrm{Op}(p^\sharp) \in {}^r\tilde{\Psi}^m_{1,1}(M).$$

Although we shall not use it, one can more generally define a paradifferential operator associated to each symbol $a \in C^rS^m(M)$ of order m, see [Hör97, Chapter X] or [Tay91].

By definition, for $P = \mathrm{Op}(p) \in {}^r\mathrm{Diff}^m(M)$,

$$\sigma(T_p)(x,\xi) = p^\sharp(x,\xi) \mod {}^r\tilde{S}^{m-1}_{1,1}(M) \tag{2.6}$$

and, if $r \ge 1$, viewing this principal symbol as an element in $C^rS^m_{1,1}(M)/C^{r-1}S^{m-1}_{1,1}(M)$, we recover by Lemma 2.14 the principal symbol of P (which has rough regularity)

$$\sigma(T_p) = \sigma(P) \mod C^{r-1}S^{m-1}_{1,1}(M).$$

This is particularly important for the propagation estimates as the Hamilton flow of $\sigma(T_p)$ for $|\xi|$ very large becomes asymptotically the Hamilton flow of $\sigma(P)$.

We then get a calculus, which is traditionally written as follows.

Proposition 2.16. [Bon81, Theorèmes 3.2 and 3.3], [Hör97, Theorem 10.2.4 and 10.2.5] *Let $P = \mathrm{Op}(p) \in {}^r\mathrm{Diff}^m(M)$ and $Q = \mathrm{Op}(q) \in {}^r\mathrm{Diff}^l(M)$ with $r > \max(r,l)$. Then*

- $T_{pq} - T_pT_q \in {}^{r-1}\tilde{\Psi}^{m+l-1}_{1,1}(M)$ *if* $r \ge 1$*;*
- $T_p^* - T_{\bar{p}} \in {}^{r-1}\tilde{\Psi}^{m-1}_{1,1}(M)$ *if* $r \ge 1$.

Proof. Using Proposition 2.8, the proof reduces to showing that $\sigma(T_{pq}) = \sigma(T_p)\sigma(T_q)$ and $\sigma(T_p^*) = \sigma(T_{\bar{p}})$. Since $\sigma(T_pT_q) = \sigma(T_p)\sigma(T_q) = p^\sharp q^\sharp \in {}^r\tilde{S}^{m+l}_{1,1}(M)/{}^{r-1}\tilde{S}^{m+l-1}_{1,1}(M)$ by Proposition 2.9, this reduces to showing that $(pq)^\sharp = p^\sharp q^\sharp$ modulo ${}^{r-1}\tilde{S}^{m+l-1}_{1,1}(M)$. In view of our regularization definition, this fact reduces in local charts to the case in $\mathbb{R}^n$, which is proved in [Hör97, Theorem 10.2.5]. The second statement is similar. □

One of the main properties of the paradifferential operators in our setting is the decomposition of products of non-smooth functions, called *paraproducts*. The idea, for a, b in Hölder or Sobolev classes, is to replace ab by some paradifferential operators up to smoother terms.

Proposition 2.17. [Bon81, Theorem 2.5], [Tay91, Section 3.5], [Hör97, Theorem 10.2.8]. *If a and b are L^∞ functions on M, then*

$$ab = T_a b + T_b a + R(a, b),$$

where the bilinear symmetric operator R as the following mapping properties:

- $R : C^r(M) \times H^s(M) \to H^{s+r}(M)$ *if* $s > 0, r > 0$,
- $R : H^s(M) \times H^t(M) \to H^{s+t-\frac{n}{2}}(M)$ *if* $s + t > n/2$,
- $R : C^r(M) \times C^\rho(M) \to C^{r+\rho}(M)$ *if* $r, \rho > 0$ *and* $r + \rho$ *is not an integer.*

In addition, if $a \in C^r(M)$, then $R_a := R(a, \cdot) \in \Psi^{-r}_{1,1}(M)$.

Proof. First we recall the results of [Bon81, Theorem 2.5], [Tay91, Section 3.5] in $\mathbb{R}^n$: $a_j b_j = \mathrm{Op}_{\mathbb{R}^n}(a_j^\sharp) b_j + \mathrm{Op}_{\mathbb{R}^n}(b_j^\sharp) a_j + R_{\mathbb{R}^n}(a_j, b_j)$ for any a_j, b_j with compact support in $\mathbb{R}^n$ and $R_{\mathbb{R}^n}$ is bounded as claimed in the Proposition. It then suffices to write $ab = \sum_j \psi_j(\tilde{\psi}_j a)(\tilde{\psi}_j b)$, and we deduce, with $a_j = \tilde{\psi}_j a$ and $b_j = \tilde{\psi}_j b$, that

$$\begin{aligned} ab =& \sum_j \psi_j a_j b_j = \sum_j \mathrm{Op}_{\mathbb{R}^n}(a_j^\sharp)\psi_j b + \mathrm{Op}_{\mathbb{R}^n}(b_j^\sharp)\psi_j a + \psi_j R_{\mathbb{R}^n}(a_j, b_j) \\ =& T_a b + T_b a + \sum_j \psi_j R_{\mathbb{R}^n}(a_j, b_j) \end{aligned}$$

by using $\psi_j b_j = \psi_j b$ and $\psi_j a = \psi_j a_j$. The fact that $R_a \in \Psi^{-r}_{1,1}(M)$ is proved in [Hör97, Theorem 10.2.8] . □

For a distribution $u \in C^{-\infty}(M)$, denote by $\mathrm{WF}_{H^s}(u)$ the complement in T^*M of those (x_0, ξ_0) such that there is a conic (in ξ) neighborhood $U \subset T^*M$ of (x_0, ξ_0) such that for all $A \in \Psi^0(M)$ with $\mathrm{WF}(A) \subset U$, one has $Au \in H^s(M)$. Note that $\mathrm{WF}(u) = \mathrm{WF}_{C^\infty}(u)$ and that, as usual with wavefront sets, using a partition of unity, cutoffs functions with small supports in charts, one can reduce the analysis to $\mathbb{R}^n$ where $A = a(D)$ are Fourier multipliers.

Recall the well-known formula $\mathrm{WF}(ab) \subset (\mathrm{WF}(a) + \mathrm{WF}(b))$. The paraproduct decomposition induces a similar decomposition of the wavefront set, so that the wavefront set of $R(a, b)$ is included in $(\mathrm{WF}(a) + \mathrm{WF}(b))$. More specifically, we have the following result.

Proposition 2.18. *Let $\varepsilon \in (0, 1), \alpha > 0, \beta > 0, \delta > 0$, and assume that $a, b \in C^\varepsilon(M)$ and $a \in H^\alpha(M)$ and $b \in H^\beta(M)$. Then one has*

$$\mathrm{WF}_{H^{\min(\alpha,\beta)+\delta+\varepsilon}}(R(a, b)) \subset \mathrm{WF}_{H^{\alpha+\delta}}(a) + \mathrm{WF}_{H^{\beta+\delta}}(b).$$

Proof. The result on M can be classically deduced from the corresponding result in $\mathbb{R}^n$ by using charts and a partition of unity. Then by taking the supports small enough we can assume that we have closed cones K_a and K_b in frequency space such that $\mathrm{WF}_{H^{\alpha+\delta}}(a) \subset \mathbb{R}^n \times K_a$ and $\mathrm{WF}_{H^{\beta+\delta}}(b) \subset \mathbb{R}^n \times K_b$ and prove that $R(a,b) \in H^{\min(\alpha,\beta)+\delta+\varepsilon}(\mathbb{R}^n)$ away from $\mathbb{R}^n \times (K_a + K_b)$. We shall use the definition of T_a using the Littlewood-Paley decomposition

With those reductions in mind, we recall that we can use Littlewood-Paley decomposition to write $a = \sum_{j=-1}^{\infty} a_j$, with $\hat{a}_{-1}$ supported near 0, and each other $\hat{a}_j$ supported in the dyadic annulus $\mathcal{C}_j = 2^j \mathcal{C}$ where the annulus $\mathcal{C}$ around the unit sphere is such that each $\mathcal{C}_j$ intersects only $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$ (hat denotes Fourier transform). Let $\rho_a \in C^\infty(\mathbb{R}^n)$ (resp. $\rho_b \in C^\infty(\mathbb{R}^n)$) be a smooth function, homogeneous of degree 0 for large ξ, equal to 1 near $K_a \cap \{\xi \geq 1\}$ (resp. $K_b \cap \{|\xi| \geq 1\}$) and 0 away from a conic neighborhood of K_a (resp. K_b). Since $\mathrm{WF}_{H^{\alpha+\delta}}(a) \subset \mathbb{R}^n \times K_a$, the sequence $2^{j(\alpha+\delta)} \left\|(1-\rho_a(D))a_j\right\|_{L^2}$ is in $\ell^2(\mathbb{N})$ (here $\rho_a(D)$ means the Fourier multiplier by $\rho_a(\xi)$). Similarly, writting $b = \sum_{j=0}^{\infty} b_j$ for the Littlewood-Paley decomposition of b, $\mathrm{WF}(b) \subset \mathbb{R}^n \times K_b$ implies that the sequence $2^{j(\beta+\delta)} \left\|(1-\rho_b(D))b_j\right\|_{L^2}$ is in $\ell^2(\mathbb{N})$. The statement $a \in C^\varepsilon$ is equivalent to saying that the sequence $2^{j\varepsilon} \left\|a_j\right\|_{L^\infty}$ is in $\ell^\infty(\mathbb{N})$, with equivalence of the norms.

Now, using the Littlewood-Paley definition $T_a u := \sum_{k=-1}^{\infty} \sum_{j=-1}^{k-1} a_j u_k$ for the paradifferential operator, one can write $R = R(a,b) = \sum_{|j-k|<2} a_j b_k$. We see immediately from support considerations that the ℓ-th Littlewood-Paley block of R, written R_ℓ, is a sum of $a_j b_k$ with $|j-k| < 2$ and $j > \ell - N_0$ for some fixed N_0. We choose a cut-off function χ_R, homogeneous of degree 0 for $|\xi| \geq 1$, supported away from $(K_a + K_b)$, and we look at the L^2 norm of $\chi_R(D) a_j b_k$ for such a piece $a_j b_k$, which is

$$\begin{aligned}\chi_R(a_j b_k) = &\chi_R(D)((\rho_a(D)a_j)(\rho_b b_k)) + \chi_R(D)(((1-\rho_a(D))a_j)(\rho_b(D)b_k)) \\ &+ \chi_R(D)((\rho_a(D)a_j)((1-\rho_b(D))b_k)) + \chi_R(D)(((1-\rho_a(D))a_j)((1-\rho_b(D))b_k)).\end{aligned}$$

The first term is zero since its Fourier transform is $\chi_R(\widehat{\rho_a a_j} \star \widehat{\rho_b b_k})$ and the support of the convolution is in $K_a + K_b$.

For the second term, there is some uniform constant $C > 0$ such that

$$\begin{aligned}&\left\|\chi_R(D)(((1-\rho_a(D))a_j)(\rho_b(D)b_k))\right\|_{L^2} \leq \left\|((1-\rho_a(D))a_j)\right\|_{L^2} \left\|b_k\right\|_{L^\infty} \\ &\leq C 2^{-j(\alpha+\delta+\varepsilon)} \left(2^{j(\alpha+\delta)} \left\|((1-\rho_a(D))a_j)\right\|_{L^2} \times 2^{j\varepsilon} \left\|b_k\right\|_{L^\infty}\right) \\ &\leq C 2^{-j(\alpha+\delta+\varepsilon)} \left(2^{j(\alpha+\delta)} \left\|((1-\rho_a(D))a_j)\right\|_{L^2} \times 2^{k\varepsilon} \left\|b_k\right\|_{L^\infty}\right)\end{aligned}$$

where we have used that multipliers of degree 0 are bounded on L^2, $\left\|\rho_b(D) b_k\right\|_{L^\infty} \leq C \left\|b_k\right\|_{L^\infty}$ with a constant C independent of k (this holds since the Fourier support of b_k is in an annulus and ρ_b is homogeneous of degree 0 for large ξ) and, at the last line, that k and j are comparable.

For the third term, we have similarly

$$\begin{aligned}
&\|\chi_R(D)((\rho_a(D)a_j)((1-\rho_b(D))b_k))\|_{L^2} \le C\,\|a_j\|_{L^\infty}\,\|(1-\rho_b(D))b_k\|_{L^2} \\
&\le C2^{j(\beta+\delta+\varepsilon)}\left(2^{j\varepsilon}\,\|a_j\|_{L^\infty} \times 2^{j(\beta+\varepsilon)}\,\|(1-\rho_b(D))b_k\|_{L^2}\right) \\
&\le C2^{-j(\beta+\delta+\varepsilon)}\left(2^{j\varepsilon}\,\|a_j\|_{L^\infty} \times 2^{k(\beta+\delta)}\,\|(1-\rho_b(D))b_k\|_{L^2}\right).
\end{aligned}$$

The bounds on the last term are proved in the same way. In the end, we find

$$2^{\ell(\min(\alpha,\beta)+\varepsilon)}\,\|\chi_R(D)R_\ell\|_{L^2} \le C \sum_{j>\ell-N_0,|\nu|\le 1} 2^{-(j-\ell)(\min(\alpha,\beta)+\delta+\varepsilon)} A_j B_{j-\nu}$$

where A_j is an ℓ^∞ sequence and B_k is an ℓ^2 sequence. Writing $\gamma := \min(\alpha,\beta)+\delta+\varepsilon$, we have by Cauchy-Schwarz inequality

$$\sum_{\ell=0}^{\infty} 2^{\ell(\min(\alpha,\beta)+\varepsilon)}\,\|\chi_R(D)R_\ell\|_{L^2}^2 \le C\sum_{\ell=0}^{\infty}\sum_{k=-N_0,|\nu|\le 1}^{\infty} 2^{-k\gamma}B_{\ell+k-\nu}^2 \le C\sum_{k\ge 0} B_k^2 < \infty$$

The result is proved. $\square$

2.3. **Propagation estimates.** In this section, we will show that the usual propagation of singularity estimates for classical pseudo-differential operators $P \in \Psi^m(M)$ apply to paradifferential operators, as well as the radial type estimates (source and sink). The proofs are essentially the same for paradifferential operators once we have the material of the previous section. For convenience of the reader we shall give some details.

First, we come back to the notion of principal symbol for T_p if $P \in {}^r\mathrm{Diff}^m(M)$. It is convenient to consider the fiber radial-compactification $\overline{T}^*M$ of the cotangent bundle, see [DZ19, Section E.1.3] for details. It amounts to adding the sphere bundle $S^*M = T^*M\setminus\{0\}/\mathbb{R}^+$ at infinity in ξ, and it becomes a ball-bundle with boundary $\partial\overline{T}^*M = S^*M$. If the principal symbol $\sigma(A)$ of an operator $A \in \Psi^m(M)$ is homogeneous of degree m, it can then be viewed as a function $(|\xi|^{-m}\sigma(A))|_{\partial\overline{T}^*M}$. Similarly, if $P \in {}^r\mathrm{Diff}^m(M)$, one recovers the principal symbol of T_p modulo $C^{r-m}S^{m-\min(1,r)}_{1,1}(M)$:

$$\sigma(T_p) = (|\xi|^{-m}\sigma(T_p))|_{\partial\overline{T}^*M} = (|\xi|^{-m}\sigma(P))|_{\partial\overline{T}^*M}.$$

Moreover, we notice that if $r>1$, $m=1$ and the principal symbol $p_1(x,\xi) := \sigma(P)(x,\xi)$ is real-valued, then P can be written as $P = -iX+V$ where X is a C^r-vector field defined by $p_1(x,\xi) = \xi(X(x))$ and where $V \in C^{r-1}(M)$ a potential. Moreover the flow φ_t of X is well-defined and wih regularity $C^r(M,M)$. As explained in [CG20, 1st paragraph of Section 4.2], the Hamilton flow H_{p_1} of $p_1(x,\xi)$ is also well-defined and given by the symplectic lift

$$\Phi_t(x,\xi) = e^{tH_{p_1}}(x,\xi) = (\varphi_t(x), (d\varphi_t(x)^{-1})^T\xi).$$

Since p_1 is homogeneous of degree 1, then H_{p_1} is homogeneous of degree 0 and it extends, as well as its flow, as a C^{r-1} vector field and flow on $\overline{T}^*M$. Note also that $H_{p_1}f = Xf$ if f is independent of ξ (i.e. the pull-back of a function on M).

We start with a technical Lemma.

Lemma 2.19. *Let H_{p_1} be the Hamilton vector field of $p_1(x,\xi)=\xi(X(x))$ on T^*M with $X\in C^r(M;TM)$ for $r>1$. Let $u\in C^{r-1}(T^*M)$, smooth in the ξ variable in the sense that $\partial_\xi^\beta u\in C^{r-1}(T^*M)$ for all β, and homogeneous of degree 0 in ξ for $|\xi|$ large. If $H_{p_1}u\in C^{r-1}(T^*M)$, then for all $\varepsilon>0$ there exists $v\in S^0(M)$, homogeneous of degree 0 in ξ for $|\xi|$ large, so that $\|u-v\|_{L^\infty}<\varepsilon$ and $\|H_{p_1}(u-v)\|_{L^\infty}<\varepsilon$. Moreover, if $u\geq 0$, we can choose $v\geq 0$.*

Proof. First, using a partition of unity, remark that it suffices to assume that u is supported in T^*U_j where $U_j\subset M$ is a chart. Then, observe that in local coordinates in the chart $H_{p_1}=X(x)-\sum_k\partial_{x_k}(\xi(X(x)))\partial_{\xi_k}$ if $p_1(x,\xi)=\xi(X(x))$ for $X\in C^r(M;TM)$. Since u and $H_{p_1}u$ are homogeneous of degree 0 for $|\xi|$ large, it suffices to consider the bound for $|\xi|$ bounded, provided we take v homogeneous of degree 0 for $|\xi|$ large. We set, after identifying $U_j\simeq B(0,1)\subset\mathbb{R}^n$, for $\chi\in C_c^\infty(B(0,1);[0,1])$ with $\int\chi=1$

$$R_\varepsilon u(x,\xi):=\varepsilon^{-n}\int_{\mathbb{R}^n}\chi(\frac{x-y}{\varepsilon})u(y,\xi)dy.$$

It is a routine exercise to see that $\|R_\varepsilon u-u\|_{L^\infty}=\mathcal{O}(\varepsilon^s\|u\|_{C^s})$ if $u\in C_c^s(B(0,1)\times K)$ for $s\in(0,1)$, where $K\subset\mathbb{R}^n$ is a compact set (in the ξ variable). Moreover $R_\varepsilon u\geq 0$ if $u\geq 0$ and R_ε commutes with all ∂_{ξ_k}. We have $X(R_\varepsilon-1)=(R_\varepsilon-1)X+B_\varepsilon$ where $B_\varepsilon:=[X^\sharp,R_\varepsilon]$ and $\|(R_\varepsilon-1)Xu\|_{L^\infty}=\mathcal{O}(\varepsilon^{r-1}\|Xu\|_{C^{r-1}})$ by using that $Xu\in C^{r-1}$. Let us then study B_ε. Writing $X(x)=\sum_k X_k(x)\partial_{x_k}$ and using that $X_k\in C^r(B(0,1))$, we get for each $f\in C^\infty$

$$\begin{aligned}B_\varepsilon f(x,\xi)=&\sum_k\varepsilon^{-n}\int_{\mathbb{R}^n}\Big(\chi(\frac{x-y}{\varepsilon})\partial_{y_k}X_k(y)+(\partial_{y_k}\chi)(\frac{x-y}{\varepsilon})\frac{X_k(x)-X_k(y)}{\varepsilon}\Big)f(y,\xi)dy\\ =&\sum_k\partial_{x_k}X_k(x)f(x,\xi)+\sum_j\partial_{x_j}X_k(x)\int\partial_{z_k}\chi(z)z_jf(x-\varepsilon z,\xi)dz+\mathcal{O}(\varepsilon^{r-1}\|f\|_{C^{r-1}})\\ =&\mathcal{O}(\varepsilon^{r-1}\|f\|_{C^{r-1}}),\end{aligned}$$

where we used $R_\varepsilon f=f+\mathcal{O}(\varepsilon^{r-1}\|f\|_{C^{r-1}})$ and $(\partial_{x_k}X_k)(y)=(\partial_{x_k}X_k)(x)+\mathcal{O}(\varepsilon^{r-1})$ for $|x-y|<\varepsilon$ in the second line, and $f(x-\varepsilon z,\xi)=f(x)+\mathcal{O}(\varepsilon^{r-1}\|f\|_{C^{r-1}})$ together with integration by parts in the third line. We conclude that $\|H_{p_1}(u-R_\varepsilon u)\|_{L^\infty}=\mathcal{O}(\varepsilon^{r-1})$ and we obtain the desired result by setting $v:=R_{\delta(\varepsilon)}u$ for $\delta(\varepsilon)=\varepsilon^{\frac{1}{r-1}}$ with $\varepsilon>0$ small. ☐

Bony proved in [Bon81, Theorem 6.2] a propagation of singularity result for real principal type operators $T_p\in{}^r\tilde\Psi^m_{1,1}(M)$ with $r>1$ and $P=\mathrm{Op}(p)\in{}^r\mathrm{Diff}^m(M)$. We recall the proof for the reader's convenience for the case $m=1$ (which is ouf case of interest).

Proposition 2.20. [Bon81, Theorem 6.2 and 6.2'] *Let $P^\sharp:=T_p\in{}^r\tilde\Psi^1_{1,1}(M)$, with $r\in(1,2)$ and $P=\mathrm{Op}(p)\in{}^r\mathrm{Diff}^1(M)$ and assume that $p_1:=\sigma(P)$ is real-valued. Let*

$$\Phi_t=\exp(tH_{p_1}):T^*M\to T^*M$$

be the flow of the Hamilton vector field of p. Let A, B, B_1 in $\Psi^0(M)$, such that for $(x,\xi)\in\mathrm{WF}(A)$, there is some $T\geq 0$ such that

$$\Phi_{-T}(x,\xi)\in\mathrm{ell}(B);\quad \Phi_t(x,\xi)\in\mathrm{ell}(B_1)\ \textit{for all}\ t\in[-T,0]. \tag{2.7}$$

Then for all $s \in \mathbb{R}$, all $\varepsilon > 0$, there is $C > 0$ such that for all $u \in H^{s-(r-1)+\varepsilon}(M)$ with $B_1 f \in H^s(M)$ and $Bu \in H^s(M)$ where $P^\sharp u = f$, then $Au \in H^s(M)$ and

$$\|Au\|_{H^s(M)} \le C\,\|Bu\|_{H^s(M)} + C\,\|B_1 f\|_{H^s(M)} + C\,\|u\|_{H^{s-(r-1)+\varepsilon}(M)}. \tag{2.8}$$

Proof. We follow the proof in [DZ19, Theorem E.47] given for operators in $\Psi^m(M)$ and will focus mainly on the differences due to our assumptions. Without loss of generality, we may assume that B_1 has non-negative principal symbol, is self-adjoint and $\mathrm{WF}(1-B_1) \cap \cup_{t=0}^{T} \Phi_{-t}(\mathrm{WF}(A)) = \emptyset$. As explained above, the Hamilton field H_{p_1} and its flow Φ_t extend to $\overline{T}^*M$ in a C^{r-1} fashion. By [Bon81, Lemme 6.6], for each $\beta \ge 0$, there is $g \in C^\infty(\overline{T}^*M, \mathbb{R}^+)$ homogeneous of degree 0 such that $\mathrm{supp}(g) \subset \mathrm{ell}(B_1)$ with $g > 0$ on $\mathrm{WF}(A)$ and $H_p g \le -\beta g$ on $\overline{T}^*M \setminus \mathrm{ell}(B)$. Set $G = \mathrm{Op}(\langle \xi \rangle^s g) \in \Psi^s(M)$ with $\mathrm{WF}(G) \subset \mathrm{ell}(B_1)$. Take $u \in C^\infty(M)$, $f = P^\sharp u$ and write

$$\mathrm{Im}\langle f, G^*Gu\rangle = \mathrm{Im}\langle \mathrm{Re}(P^\sharp)u, G^*Gu\rangle + \mathrm{Re}\langle \mathrm{Im}(P^\sharp)u, G^*Gu\rangle$$

The first term in the RHS can be written as $\langle Zu, u\rangle$ where $Z := \frac{i}{2}[\mathrm{Re}(P^\sharp), G^*G]$. Since $\mathrm{Re}(P^\sharp) \in {}^{r}\tilde{\Psi}^1_{1,1}(M)$ with $r > 1$, Proposition 2.9 insures that $Z \in {}^{r-1}\tilde{\Psi}^{2s}_{1,1}(M)$. Moreover, using Proposition 2.9 and (2.6), Z has principal symbol

$$\sigma(Z) = \langle \xi \rangle^{2s}(gH_{p_1}g + sH_{p_1}(\log\langle \xi \rangle)g^2) + o(|\xi|^{2s})$$

as $|\xi| \to \infty$. Thus there is $C_1 > 0$ such that $\sigma(Z + (\beta - C_1)G^*G) \le 0$ near $\overline{T}^*M \setminus \mathrm{ell}(B)$ for $|\xi|$ large enough, and by the sharp Gårding inequality (Proposition 2.12) applied to $B_1 Z B_1$, then for each $N > 0$ there is $C > 0$ so that

$$\langle ZB_1u, B_1u\rangle \le (C_1 - \beta)\|Gu\|^2_{L^2(M)} + C\|Bu\|^2_{H^s(M)} + C\|B_1u\|^2_{H^{s-\frac{r-1}{4}}(M)} + C\|u\|_{H^{-N}(M)}.$$

Note that $(1-B_1)Z(1-B_1) \in \Psi^{-\infty}(M)$, $(1-B_1)ZB_1 \in \tilde{\Psi}^{2s-r-1}_{1,1}(M)$ and $B_1Z(1-B_1) \in \tilde{\Psi}^{2s-(r-1)}_{1,1}(M)$ using (2.3), thus for each $\varepsilon_0 \in (0, (r-1)/2)$

$$|\langle (Z - B_1ZB_1)u, u\rangle| \le C\|B_1u\|^2_{H^{s-\varepsilon_0}(M)} + C\|u\|^2_{H^{s-(r-1)+\varepsilon_0}(M)}.$$

Next, we deal with $\langle \mathrm{Im}(P)u, G^*Gu\rangle$: first we have

$$|\langle \mathrm{Im}(P^\sharp)u, G^*Gu\rangle| \le |\langle \mathrm{Im}(P^\sharp)Gu, Gu\rangle| + \langle G^*[G, \mathrm{Im}(P^\sharp)]u, u\rangle|.$$

Using that $p_1 = \sigma(P^\sharp)$ is real, we have $\mathrm{Im}(P^\sharp) \in {}^{r-1}\tilde{\Psi}^0_{1,1}(M)$ by Proposition 2.16, and using Proposition 2.9 we get $G^*[G, \mathrm{Im}(P^\sharp)] \in \tilde{\Psi}^{2s-(r-1)}_{1,1}(M)$. Together with Proposition 2.6 and the fact that $\mathrm{WF}(G) \subset \mathrm{ell}(B_1)$, we obtain for $\varepsilon_0 > 0$ as above

$$|\langle \mathrm{Im}(P^\sharp)u, G^*Gu\rangle| \le C_2\|Gu\|^2_{L^2(M)} + C\|B_1u\|^2_{H^{s-\varepsilon_0}(M)} + C\|u\|^2_{H^{s-(r-1)+\varepsilon_0}(M)}$$

for some constant $C_2 > 0, C > 0$. Therefore

$$|\mathrm{Im}\langle f, G^*Gu\rangle| \le (C_1 + C_2 - \beta)\|Gu\|^2_{L^2(M)} + C\|Bu\|^2_{H^s(M)} + C\|B_1u\|^2_{H^{s-\varepsilon_0}(M)} + C\|u\|^2_{H^{s-(r-1)+\varepsilon_0}(M)}$$

Taking $\beta = C_1 + C_2 + 1$, and using $|\mathrm{Im}\langle f, G^*Gu\rangle| \le C\|B_1f\|^2_{H^s} + \|Gu\|^2_{L^2}$ we deduce that

$$\|Gu\|^2_{L^2} \le C\|B_1f\|^2_{H^s} + C\|Bu\|^2_{H^s} + C\|B_1u\|^2_{H^{s-\varepsilon_0}} + C\|u\|^2_{H^{s-(r-1)+\varepsilon_0}}. \tag{2.9}$$

Since $\|Au\|_{H^s(M)} \leq C\|Gu\|_{L^2}$ for some $C > 0$ by elliptic estimates, we get

$$\|Au\|_{L^2}^2 \leq C\|B_1 f\|_{H^s}^2 + C\|Bu\|_{H^s}^2 + C\|B_1 u\|_{H^{s-\varepsilon_0}}^2 + C\|u\|_{H^{s-(r-1)+\varepsilon_0}}^2.$$

We can then iterate this argument just as step 5 in the proof of [DZ19, Theorem E.47], and improve the $\|B_1 u\|_{H^{s-\varepsilon_0}}^2$ term to $\|B_1 u\|_{H^{s-\ell\varepsilon_0}}^2$ for all ℓ large, we then obtain (2.8) for $u \in C^\infty(M)$. Using the regularization argument [DZ19, Lemma E.45], (2.8) also holds for $u \in H^s(M)$ so that $P^\sharp u \in H^s(M)$. To extend to general $u \in H^{s-(r-1)+\varepsilon_0}(M)$ with $B_1 P u \in H^s$ and $Bu \in H^s$, one can use the regularization procedure explained in [DZ19, Section E.7, Exercices 10 and 31] which adapts exactly in the same way to our setting: set $N = s + (r-1) + \varepsilon_0$, we use the regularization operator $X_\epsilon := \mathrm{Op}(\langle \epsilon\xi\rangle^{-N})$ and an elliptic approximate inverse Y_ϵ so that $X_\varepsilon Y_\varepsilon = 1 + \mathcal{O}_{\Psi^{-\infty}(M)}(\varepsilon^\infty)$; as operators in $\Psi^0(M)$, they are bounded uniformly in ϵ; for $r > 0$, one can use Proposition 2.9 to get

$$P_\epsilon^\sharp := X_\epsilon P^\sharp Y_\epsilon = P^\sharp + i\mathrm{Op}(\langle \epsilon\xi\rangle^N \{p_1, \langle \epsilon\xi\rangle^{-N}\}) + \mathcal{O}_{\tilde{\Psi}_{1,1}^{-(r-1)}}(1), \tag{2.10}$$

as in the classical case. The symbol $\langle \epsilon\xi\rangle^N \{p_1, \langle \epsilon\xi\rangle^{-N})\}$ is uniformly bounded in $S_{1,1}^0(M)$, the region in $\overline{T}^*M$ where $P_\varepsilon^\sharp$ and $P^\sharp$ are not microlocally in $\tilde{\Psi}_{1,1}^{1-r}(M)$ are the same (thus uniform in ε), and $u \in H^{-N}(M)$ is in $H^s(M)$ if $X_\epsilon u$ is uniformly bounded in $H^s(M)$. Using this, the proof above extends to give

$$\|X_\varepsilon A u\|_{L^2}^2 \leq C\|X_\varepsilon B_1 f\|_{H^s}^2 + C\|X_\varepsilon B u\|_{H^s}^2 + C\|u\|_{H^{-N}}^2.$$

for all $u \in H^{-N}(M)$, uniformly in $\varepsilon > 0$, and let $\varepsilon \to 0$ to obtain the result. □

Remark 1. This result also extends in the obvious way if $P \in {}^r\mathrm{Diff}^1(M; E)$ acts on a bundle E and the principal symbol of P is of the form $p_1 \otimes \mathrm{Id}$ for some $p_1 \in C^r(T^*M)$ as above.

The second type of propagation estimates are the radial estimates. These were introduced by Melrose in scattering theory [Mel94] and developped by Dyatlov-Zworski [DZ16, DZ19] for smooth Anosov flows. We will now explain how to modify their proof to adapt them to paradifferential operators. We notice that for smooth generators of hyperbolic flows, radial source/sink estimates are also proved by Guedes Bonthonneau-Lefeuvre in Hölder spaces [GBL].

We assume that P is as in Proposition 2.20. The principal symbol will be $\sigma(P) = p_1$, real valued and homogeneous of degree 1, and we denote by $\Phi_t = e^{tH_{p_1}}$ the Hamilton flow acting on the fiber-radial compactification of the cotangent bundle $\overline{T}^*M$.

A *radial source* is a nonempty compact Φ_t-invariant set

$$L \subset \left\{\langle \xi\rangle^{-1} p = 0\right\} \cap \partial\overline{T}^*M,$$

such that for some neighborhood $U \subset \overline{T}^*M$ of L, one has for all $(x, \xi) \in U$,

$$\kappa(\phi_t(x,\xi)) \to L, \quad |\Phi_t(x,\xi)| \geq Ce^{\nu|t|}|\xi|$$

for some $C, \nu > 0$ and all $t \leq 0$; here $\kappa : T^*M \to \partial\overline{T}^*M$ is the canonical projection. A *radial sink* is a radial source for $-p_1$.

A function $a \in C^0(\overline{T}^*M)$ is called *eventually positive on* L if $\exists T > 0$ such that

$$\int_0^T a \circ \Phi_t \, \mathrm{d}t > 0 \quad \text{on } L.$$

A symbol is said eventually negative if $-a$ is eventually positive.

Lemma 2.21. *Let P is as in Proposition 2.20 with principal symbol $\sigma(P) = p_1$. Let $a \in {}^s\tilde{S}^0_{1,1}(M)$ for $s \in (0,1)$ such that that there exists $a_0 \in C^s S^0(M)$, homogeneous of degree 0, with $a - a_0 \in C^s S^{-s}_{1,1}(M)$. Assume that a_0, and thus a, is eventually positive on the radial source L. Then, there exists $b \in S^0(M)$ homogeneous of degree 0 for large $|\xi|$ such that $H_{p_1} b + a > 0$ on L.*

Proof. We can take a smooth $a_0' \in S^0(M)$ so that $\|a_0' - a_0\|_{C^0} < \varepsilon$ for $\varepsilon > 0$ small. Let us then define in the region $\{|\xi| > 1\}$

$$b_0 := \frac{1}{T} \int_0^T (T - t) a_0' \circ \Phi_t \, \mathrm{d}t \in C^{r-1} S^0(M) \tag{2.11}$$

which is homogeneous of degree 0, and we have $H_{p_1} b_0 + a_0' = \frac{1}{T} \int_0^T a_0' \circ \Phi_t \, \mathrm{d}t \in C^{r-1} S^0(M)$ which is well-defined in $\{|\xi| > 1\}$. Moreover one has on L

$$a_0 + H_{p_1} b_0 > a_0' + H_{p_1} b_0 - \varepsilon > \frac{1}{T} \int_0^T a_0 \circ \Phi_t \, \mathrm{d}t - 2\varepsilon > \varepsilon$$

if $\varepsilon < \frac{1}{3T} \int_0^T a_0 \circ \Phi_t \, \mathrm{d}t$. Using Lemma 2.19, there is $b \in S^0(M)$ so that $\|H_{p_1} b - H_{p_1} b_0\|_{L^\infty} < \varepsilon$, which ends the proof. $\square$

Proposition 2.22. *Let $P = \mathrm{Op}(p) \in {}^r\mathrm{Diff}^1(M)$ with $r > 1$ and assume that $p_1 := \sigma(P)$ is real valued. Let $\Phi_t = e^{tH_{p_1}} : T^*M \to T^*M$ be the flow of the Hamilton vector field of p_1 and assume $L \subset \partial\overline{T}^*M$ is a radial source for p_1. Let $P^\sharp := T_p \in {}^r\tilde{\Psi}^1_{1,1}(M)$ be its paradifferential operator. Let $s_0 \in \mathbb{R}$ satisfying the following* threshold condition*:*

$$\sigma(\operatorname{Im} P) + s_0 \frac{H_{p_1} \langle \xi \rangle}{\langle \xi \rangle} \text{ is eventually negative on } L. \tag{2.12}$$

If $s \geq s_0$, then for any $B_1 \in \Psi^0(M)$ such that $L \subset \mathrm{ell}(B_1)$, there exists $A \in \Psi^0(M)$ with $L \subset \mathrm{ell}(A)$, such that for all $\varepsilon > 0$ there is $C > 0$ such that for all $u \in H^{s-(r-1)+\varepsilon}(M)$ with $B_1 u \in H^{s_0}(M)$, $B_1 P^\sharp u \in H^s(M)$, then $Au \in H^s(M)$ and

$$\|Au\|_{H^s(M)} \leq C \|B_1 P^\sharp u\|_{H^s(M)} + C \|u\|_{H^{s-(r-1)+\varepsilon}(M)}. \tag{2.13}$$

Proof. Now that we have all the paradifferential calculus properties at hand, the proof is essentially the same as [DZ19, Theorem E.52]. We first prove (2.13) for $u \in C^\infty(M)$. Using that L is a source, observe that, if (2.12) is true, then it is also true by replacing s_0 by any $s \geq s_0$. Notice that $\operatorname{Im} P^\sharp \in {}^{r-1}\tilde{\Psi}^0_{1,1}(M)$ using Proposition 2.16, and its principal symbol $\sigma(\operatorname{Im} P^\sharp)$ can be obtained using Proposition 2.8 and the expression (2.5) for $p^\sharp$: we have

$$\sigma(\operatorname{Im} P^\sharp) - \sigma(\operatorname{Im} P) \in C^{r-1} S^{-(r-1)}_{1,1}(M).$$

Thus by Lemma 2.21 and (2.12), there is $b \in S^0(M)$ homogeneous of degree 0 for large $|\xi|$ so that $\sigma(\operatorname{Im} P^\sharp) + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} - H_{p_1}b < 0$. Take $U \subset \overline{T}^*M$ a neighborhood of L such that

$$\exists \delta > 0 \text{ small }, \quad \sigma(\operatorname{Im} P) + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} - H_{p_1}b < -2\delta \quad \text{ in } U.$$

Then, as in [DZ19, Lemma E.53], we claim that for each $\varepsilon > 0$ small, there is $\chi \in C_c^\infty(U;\mathbb{R}^+)$ homogeneous of degree 0 near $\partial\overline{T}^*M$ so that $\chi > 0$ on L and $H_{p_1}\chi \leq \varepsilon$ and $\operatorname{supp}(H_{p_1}\chi) \cap L = \emptyset$. To obtain this function, we shrink U so that $\Phi_t(x,\xi) \to L$ as $t \to -\infty$ uniformly, let $\psi \in C_c^\infty(U,[0,1])$ equal to 1 near L and homogeneous of degree 0 for $|\xi|$ large, then set $\chi_0 := T^{-1}\int_T^{2T} \psi \circ \Phi_{-t}\, dt$ where $T > 0$ is large enough so that $\Phi_{-t}(\operatorname{supp}(\psi)) \subset \{\psi = 1\}$ for all $t \geq T$. One has $\chi_0 \geq 0$, $\chi_0 = 1$ near L (thus $H_{p_1}\chi_0 = 0$ near L), χ_0 homogeneous of degree 0 for large $|\xi|$, $H_{p_1}\chi_0 \leq 0$ and $\chi_0 > 0$ on L, but χ_0 is only C^{r-1} in the x-variable. Using Lemma 2.19, we find, for any $\varepsilon > 0$ small, a function $\chi_\varepsilon \in S^0(M)$ non-negative so that $\chi_\varepsilon = 1$ in an ε-independent neighborhood of L, and as $\varepsilon > 0$ goes to 0, $\|\chi_\varepsilon - \chi_0\|_{C^0} \leq \varepsilon$ and $\|H_{p_1}(\chi_\varepsilon - \chi_0)\|_{L^\infty} \leq \varepsilon$. Thus $H_{p_1}(\chi_\varepsilon) \leq \varepsilon$ and $\operatorname{supp}(H_{p_1}\chi_\varepsilon)$ is contained in a fixed compact set of $\overline{T}^*M$ (uniform in $\varepsilon > 0$) not intersecting L .

We can then define $g_\varepsilon := e^{-b}\chi_\varepsilon \in C_c^\infty(U) \cap S^0(M)$ homogeneous of degree 0 for $|\xi|$ large, and remark that

$$g_\varepsilon H_{p_1} g_\varepsilon + \sigma(\operatorname{Im} P) g_\varepsilon^2 + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} g_\varepsilon^2 \leq -\delta g_\varepsilon^2 + e^{-2b}\chi_\varepsilon H_{p_1}(\chi_\varepsilon) \leq -\delta g_\varepsilon^2 + \varepsilon\tilde{\chi}^2 \tag{2.14}$$

where $\tilde{\chi} \in C_c^\infty(U,\mathbb{R}^+)$ is supported far from L and independent of $\varepsilon > 0$. Then we claim that if $A, B_2 \in \Psi^0(M)$ are such that $g > 0$ on $\mathrm{WF}(A)$, B_2 equal to 1 microlocally near L and $\operatorname{supp}(g) \cap \mathrm{WF}(1 - B_2) = \emptyset$, for all $\varepsilon_0 > 0$ small there is $C > 0$ such that for all $u \in C^\infty(M)$, if $f := P^\sharp u$,

$$\|Au\|_{H^s(M)} \leq C(\|B_2 f\|_{H^s(M)} + \|B_2 u\|_{H^{s-\varepsilon_0}(M)} + \|u\|_{H^{s-(r-1)+\varepsilon_0}(M)}) \tag{2.15}$$

To prove this, we proceed as in the proof of Proposition 2.20. Let $G_\varepsilon = \operatorname{Op}(\langle\xi\rangle^s g_\varepsilon) \in \Psi^{2s}(M)$. Then $\langle f, G_\varepsilon^* G_\varepsilon u\rangle = \operatorname{Re}\langle Zu, u\rangle$ with $Z := \frac{i}{2}[\operatorname{Re} P^\sharp, G_\varepsilon^* G_\varepsilon] + G_\varepsilon^* G_\varepsilon \operatorname{Im} P^\sharp \in {}^{r-1}\tilde{\Psi}^{2s}_{1,1}(M)$ and Z has principal symbol (using $r > 1$ and Proposition 2.9)

$$\sigma(Z) = \langle\xi\rangle^{2s}\Big(g_\varepsilon H_{p_1} g_\varepsilon + g_\varepsilon^2\Big(s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} + \sigma(\operatorname{Im} P^\sharp)\Big)\Big) + o(|\xi|^{2s}) \text{ as } |\xi| \to \infty.$$

By (2.14), $\sigma(Z + \delta G_\varepsilon^* G_\varepsilon - \varepsilon \operatorname{Op}(\tilde{\chi}\langle\xi\rangle^s)^* \operatorname{Op}(\tilde{\chi}\langle\xi\rangle^s)) \leq 0$, thus by Proposition 2.12 (as in the proof of Proposition 2.20), there is $C > 0$ (independent of ε) and $C_\varepsilon > 0$ depending on $\varepsilon > 0$ so that for each $u \in C^\infty$

$$\operatorname{Re}\langle Zu, u\rangle \leq -\delta\|G_\varepsilon u\|_{L^2}^2 + C\varepsilon\|\operatorname{Op}(\tilde{\chi})u\|_{H^s}^2 + C_\varepsilon\|B_2 u\|_{H^{s-\varepsilon_0}}^2 + C_\varepsilon\|u\|_{H^{s-(r-1)+\varepsilon_0}}^2.$$

Then as in the proof of Proposition 2.20 we get by Cauchy-Schwartz

$$\delta\|G_\varepsilon u\|_{L^2}^2 \leq C_\varepsilon\|B_2 f\|_{H^s}^2 + C\varepsilon\|\operatorname{Op}(\tilde{\chi})u\|_{H^s}^2 + C_\varepsilon\|B_2 u\|_{H^{s-\varepsilon_0}}^2 + C_\varepsilon\|u\|_{H^{s-(r-1)+\varepsilon_0}}^2$$

for some $C>0$ uniform in $\varepsilon>0$. Since $g_\varepsilon=\langle\xi\rangle^s e^{-b}$ in an ε-independent neighborhood of L, we can find $A=\mathrm{Op}(a)\in\Psi^s(M)$ independent of $\varepsilon>0$ with $a\in C_c^\infty(U)$ such that $g_\varepsilon\geq\langle\xi\rangle^s a\geq c_0\langle\xi\rangle^s$ near L for some $c_0>0$. This implies, by sharp Gårding applied to $G_\varepsilon^*G_\varepsilon-\mathrm{Op}(\langle\xi\rangle^s a)^*\mathrm{Op}(\langle\xi\rangle^s a)$, that there is $C>0$ independent of $\varepsilon>0$ and $C_\varepsilon>0$ depending on $\varepsilon>0$ such that $\|G_\varepsilon u\|^2_{L^2}\geq C^{-1}\|Au\|^2_{H^s}-C_\varepsilon\|B_2u\|^2_{H^{s-\varepsilon_0}}-C_\varepsilon\|u\|^2_{H^{s-(r-1)+\varepsilon_0}}$. Therefore we obtain

$$\delta\|Au\|^2_{H^s}\leq C_\varepsilon\|B_2f\|^2_{H^s}+C\varepsilon\|\mathrm{Op}(\tilde\chi)u\|^2_{H^s}+C_\varepsilon\|B_2u\|^2_{H^{s-\varepsilon_0}}+C_\varepsilon\|u\|^2_{H^{s-(r-1)+\varepsilon_0}}. \tag{2.16}$$

Since $\mathrm{supp}(\tilde\chi)\cap L=\emptyset$ and L is a source, there is a uniform time T' so that for each $(x,\xi)\in\mathrm{supp}(\tilde\chi)$, $\cup_{t\in[0,T]}\Phi_{-t}(x,\xi)\cap\mathrm{WF}(A)\neq\emptyset$. We can then apply Proposition 2.20 with (A,B) replaced by $(\mathrm{Op}(\tilde\chi),A)$: this gives for some $C>0$ uniform in $\varepsilon>0$

$$\|\mathrm{Op}(\tilde\chi)u\|^2_{H^s}\leq C\|Au\|^2_{H^s}+C\|B_1f\|^2_{H^s}+C\|u\|^2_{H^{s-(r-1)+\varepsilon_0}}.$$

Combining with (2.16), this leads, by fixing $\varepsilon\ll\delta$, to the bound (2.15). Then, the argument of step 2 in [DZ19, Theorem E.52], based on applying Proposition 2.20 to remove the $\|B_2u\|^2_{H^{s-\varepsilon_0}}$ term, can be applied verbatim in our case and yields the estimate (2.13) for all $u\in C^\infty(M)$. Using the regularization argument [DZ19, Lemma E.45], (2.13) also holds for $u\in H^s(M)$ so that $P^\sharp u\in H^s(M)$.

Finally, to obtain the result for $u\in H^{s-(r-1)+\varepsilon_0}(M)$ with $B_1u\in H^{s_0}(M)$, $B_1f\in H^s(M)$ set $G=\mathrm{Op}(\langle\xi\rangle^s)$, we apply the same argument as in [DZ19, Section E.7, Exercise 35]. First we regularize using $X_\epsilon=\mathrm{Op}(\langle\epsilon\xi\rangle^{-(s-s_0)})$ for $\varepsilon>0$ small and its approximate inverse Y_ϵ satisfying $X_\varepsilon Y_\varepsilon=1+\mathcal{O}_{\Psi^{-\infty}(M)}(\varepsilon^\infty)$; as operator in $\Psi^0(M)$ (resp. $\Psi^{(s-s_0)}(M)$), X_ε (resp. Y_ε) is bounded uniformly in ϵ. We have for all $\varepsilon>0$ small (as (2.10))

$$P^\sharp_\epsilon:=X_\epsilon P^\sharp Y_\epsilon=P^\sharp+i\mathrm{Op}\left(\langle\epsilon\xi\rangle^{s-s_0}H_{p_1}(\langle\epsilon\xi\rangle^{-(s-s_0)})\right)+\mathcal{O}_{\tilde\Psi^{-(r-1)}_{1,1}}(1).$$

The symbol

$$\langle\epsilon\xi\rangle^{(s-s_0)}H_{\sigma(P^\sharp)}(\langle\epsilon\xi\rangle^{-(s-s_0)})=-(s-s_0)\frac{\varepsilon^2\langle\xi\rangle^2}{\langle\varepsilon\xi\rangle^2}\frac{H_{\sigma(P^\sharp)}\langle\xi\rangle}{\langle\xi\rangle} \tag{2.17}$$

is uniformly bounded in ${}^{r-1}\tilde S^0_{1,1}(M)$ in ε, and $u\in H^{s_0}(M)$ belongs to $H^s(M)$ if $X_\epsilon u$ is uniformly bounded in $H^s(M)$. It suffices to show the estimate

$$\|X_\epsilon Au\|_{H^s(M)}\leq C\,\|X_\epsilon B_1f\|_{H^s(M)}+C\,\|u\|_{H^{s-(r-1)+\varepsilon_0}(M)}, \tag{2.18}$$

which is a consequence of proving the radial estimates for $P^\sharp_\epsilon$ and $u\in C^\infty(M)$ uniformly in $\varepsilon>0$. Using (2.17), one sees that $P^\sharp_\epsilon$ satisfies the threshold condition uniformly in $\varepsilon>0$, i.e. $\int_0^T(\sigma(\mathrm{Im}\,P^\sharp_\varepsilon)+s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle})\circ\phi_t\,dt<0$ for some $T>0$ independent of $\varepsilon>0$. The proof of (2.13) then applies for $P^\sharp_\varepsilon$ replacing $P^\sharp$ and $X_\varepsilon u$ replacing u, if with $u\in H^{s-(r-1)+\varepsilon_0}(M)$:

$$\|AX_\epsilon u\|_{H^s(M)}\leq C\,\|B_1X_\varepsilon f\|_{H^s(M)}+C\,\|X_\varepsilon u\|_{H^{s-(r-1)+\varepsilon_0}(M)}+o(1)\|u\|_{H^{s-(r-1)+\varepsilon_0}}$$

as $\varepsilon\to0$. Then one uses that, if $Q=\mathrm{Op}(q)\in\Psi^0(M)$ and $\tilde Q\in\Psi^0(M)$ elliptic on $\mathrm{WF}(Q)$,

$$\begin{aligned}QX_\varepsilon=&X_\varepsilon Q-[i\mathrm{Op}(\langle\epsilon\xi\rangle^{s-s_0}\{q,\langle\epsilon\xi\rangle^{-(s-s_0)}\})+\mathcal{O}_{\Psi^{-2}(M)}(1)]X_\varepsilon\tilde Q+\mathcal{O}_{\Psi^{-\infty}(M)}(1)\\=&X_\varepsilon Q+\mathcal{O}_{\Psi^{-1}(M)}(1)X_\varepsilon\tilde Q+\mathcal{O}_{\Psi^{-\infty}(M)}(1)\end{aligned}$$

so that for all $u \in H^{s-(r-1)+\varepsilon_0}(M)$, $|\|QX_\varepsilon u\|_{H^s(M)} - \|X_\varepsilon Qu\|_{H^s(M)}| \leq C\|X_\varepsilon \tilde{Q}u\|_{H^{s-1}(M)} + C\|u\|_{H^{s-(r-1)+\varepsilon_0}}$ for some uniform $C > 0$. Applying this to $Q = A$ and $Q = B_1$, we get for each $B_1' \in \Psi^0(M)$ elliptic near B_1

$$\|X_\epsilon Au\|_{H^s(M)} \leq C\,\|X_\varepsilon B_1' f\|_{H^s(M)} + C\|X_\varepsilon B_1 u\|_{H^{s-1}(M)} + C\,\|u\|_{H^{s-(r-1)+\varepsilon_0}(M)}$$

Then, we apply again the argument of step 2 in [DZ19, Theorem E.52] (based on applying Proposition 2.20) to remove the $\|X_\varepsilon B_1 u\|_{H^{s-1}}$ term, and we finally reach the estimate (2.18) with B_1' instead of B_1. □

Remark 2. Here again, the result also holds for operators acting on bundles, provided the principal symbol is of the form $\sigma(\operatorname{Re} P)(x,\xi) = p_1(x,\xi) \otimes \mathrm{Id}$

In the same way, we can prove the following:

Proposition 2.23. *Let $P = \mathrm{Op}(p) \in {}^r\mathrm{Diff}^1(M)$ with $r > 1$ and assume that $p_1 := \sigma(P)$ is real-valued. Let $\Phi_t = e^{tH_{p_1}} : T^*M \to T^*M$ be the flow of the Hamilton vector field of p_1 and assume $L \subset \partial\overline{T}^*M$ is a radial sink for p_1. Let $P^\sharp := T_p \in {}^r\tilde{\Psi}^1_{1,1}(M)$ be its paradifferential operator. Assume that there exists $s_0 \in \mathbb{R}$ satisfying the following* threshold condition*:*

$$\sigma(\operatorname{Im} P) + s_0 \frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} \text{ is eventually negative on } L.$$

Then for any $B_1 \in \Psi^0(M)$ such that $L \subset \mathrm{ell}(B_1)$, there exists $A, B \in \Psi^0(M)$ such that $L \subset \mathrm{ell}(A)$, $\mathrm{WF}(B) \subset \mathrm{ell}(B_1) \setminus L$, such that for all $\varepsilon_0 > 0$, there is $C > 0$ so that for all $u \in H^{s-(r-1)+\varepsilon_0}(M)$ such that $Bu \in H^s(M)$ and $B_1 P^\sharp u \in H^s(M)$, then $Au \in H^s(M)$ with

$$\|Au\|_{H^s(M)} \leq C\|Bu\|_{H^s(M)} + C\left\|B_1 P^\sharp u\right\|_{H^s(M)} + C\,\|u\|_{H^{s-(r-1)+\varepsilon_0}(M)}.$$

Proof. The proof follows closely the line of [DZ19, Theorem E.54]. By Lemma 2.21 there is $b \in S^0(M)$ homogeneous of degree 0 for large $|\xi|$ so that $\sigma(\operatorname{Im} P^\sharp) + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} - H_{p_1} b < 0$. Take $U \subset \overline{T}^*M$ a neighborhood of L such that $\mathrm{ell}(B_1) \subset U$ and

$$\exists \delta > 0, \quad \sigma(\operatorname{Im} P) + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} - H_{p_1} b < -2\delta \quad \text{ in } U.$$

Let $\chi_1, \chi_2 \in C_c^\infty(U)$ such that $\chi_1 = 1$ on $\mathrm{supp}(\chi_2)$ and $\chi_2 = 1$ on L, and let $\psi \in C_c^\infty(U \setminus L)$ equal to 1 on $\mathrm{supp}(\chi_1) \cap \mathrm{supp}(1-\chi_2)$. Let $g := e^{-b}\chi_1$, $A = \mathrm{Op}(\chi_2)$, $B = \mathrm{Op}(\psi)$, $B_2 = A + B$ and $G = \mathrm{Op}(\langle\xi\rangle^s g)$. Then as in Proposition 2.22, if $f := P^\sharp u$ for $u \in C^\infty(M)$, one has $\langle f, G^*Gu\rangle = \operatorname{Re}\langle Zu, u\rangle$ with $Z := \frac{i}{2}[\operatorname{Re} P^\sharp, G^*G] + G^*G \operatorname{Im} P^\sharp \in {}^{r-1}\tilde{\Psi}^{2s}_{1,1}(M)$ and

$$\sigma(Z - C_0^2 B'^* B' + \delta G^*G) = \langle\xi\rangle^{2s}(gH_{p_1}g + \sigma(\operatorname{Im} P)g^2 + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle}g^2) + o(|\xi|^{2s}) \leq 0$$

for $|\xi|$ large, where $B' := \mathrm{Op}(\langle\xi\rangle^s\psi)$ and $C_0\psi \geq H_p(\chi_1)$. Applying Proposition 2.12 to $Z - C_0^2 B'^* B' + \delta G^*G$, there is $C > 0$ such that for all $u \in C^\infty(M)$

$$\operatorname{Re}\langle Zu, u\rangle \leq -\delta\|Gu\|^2_{L^2} + C\|Bu\|^2_{H^s} + C\|B_2 u\|^2_{H^{s-\varepsilon_0}} + C\|u\|^2_{H^{s-(r-1)+\varepsilon_0}}.$$

Proceeding as in the proof of Proposition 2.20, we obtain

$$\begin{aligned}\|Au\|_{H^s} \leq & C\|Bu\|_{H^s} + C\|B_1 f\|_{H^s} + C\|B_2 u\|_{H^{s-\varepsilon_0}} + C\|u\|_{H^{s-(r-1)+\varepsilon_0}} \\ \leq & C\|Bu\|_{H^s} + C\|B_1 f\|_{H^s} + C\|Au\|_{H^{s-\varepsilon_0}} + C\|u\|_{H^{s-(r-1)+\varepsilon_0}}.\end{aligned}$$

Then an interpolation estimate, exactly as in [DZ19, Theorem E.54], allows to absorb the $\|Au\|_{H^{s-\varepsilon_0}}$ term. To obtain the result for $u \in H^{s-(r-1)+\varepsilon_0}(M)$ so that $Bu \in H^s(M)$ and $B_1 f \in H^s(M)$, it suffices to apply the same regularization procedure explained in the proof of Proposition 2.22. □

The case of operators acting on vector bundles. Let us finally discuss briefly the case of an operator $P \in {}^r\mathrm{Diff}^1(M; E)$ acting on a Hermitian vector bundle $(E, \langle\cdot,\cdot\rangle_E)$, with $P = -i\nabla_X + V$ where $V \in C^{r-1}(M; E \otimes E^*)$, ∇ a smooth connection on E and $X \in C^r(M; TM)$ a C^r vector field. The principal symbol is $\sigma(P) = p_1(x,\xi) \otimes \mathrm{Id}$ where $p_1(x,\xi) = \xi(X(x))$ is real valued, scalar, and linear in ξ. In that case, let Φ_t be the Hamilton flow of $p_1(x,\xi)$ as before. The operator P^* is defined using $\langle\cdot,\cdot\rangle_E$ and a fixed smooth measure on M. Consider the pull-back bundle $\tilde{E} := \pi^* E$ on $\overline{T}^* M$ where $\pi : \overline{T}^* M \to M$ is the projection on the base, and let $\tilde{\nabla} := \pi^*\nabla$ be the pull-back connection. An element $a \in \tilde{E} \otimes \tilde{E}^*$ is said symmetric if for $v, v' \in \tilde{E}$, $\langle a(v'), v\rangle_E = \langle v', a(v)\rangle_E$, in that case a can be identified with an element in the symmetric bundle $S^2\tilde{E}^*$.

Then Proposition 2.22 and Proposition 2.23 also hold for such P, assuming that

$$\sigma(\mathrm{Im}P) + s_0 \frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle}\mathrm{Id} < 0 \text{ on } L,$$

when we view $\sigma(\mathrm{Im}P)$ as a section of $S^2\tilde{E}^*$ using the Hermitian product $\langle\cdot,\cdot\rangle_E$. Indeed, the proof of these proposition apply verbatim to this case (taking $b = 0$ there), using that the sharp Gårding inequality also holds on bundles.

Another case of interest is when $E = (\otimes^p TM) \otimes (\otimes^q T^*M)$ is the bundle of tensors. Then the same proof applies to operators of the form $i(\mathcal{L}_X + V)$ where $\mathcal{L}_X$ is the Lie derivative and V a potential; indeed the principal symbol of $\mathcal{L}_X$ is still $p_1(x,\xi) = \xi(X)$.

3. Regularity and rigidity of the stable/unstable foliation

In this section, we will use the tools discussed in previous section to prove regularity and rigidity results on the stable/unstable foliation of Anosov flows.

3.1. **Dimension 3.** Let M be a compact 3-dimensional manifold and let X be a smooth vector field generating an Anosov flow φ_t, with Anosov splitting $\mathbb{R}X \oplus E_u \oplus E_s$. To make the arguments a bit more concrete, we will assume that E_u, E_s are trivialisable[5]. By [HPS70], the bundles E_u, E_s are $C^\gamma(M)$. More precisely, (see [FG18, Lemma 2.2]) there exist vector

[5]this assumption can easily be removed by working as in the higher dimensional case considered later, the important object being the projector $\pi_H = TM \to \mathcal{H}$ as in (3.2) where $\mathcal{H}$ is a smooth one-dimensional bundle approximating E_u

fields $U_-, U_+ \in C^\alpha(M;TM)$ for some $\alpha>0$ so that $E_u = \mathbb{R}U_-$ and $E_s=\mathbb{R}U_+$. There are $C^\alpha(M)$ functions $r_\pm$ so that

$$\begin{aligned}
[X,U_-]=-r_-U_-, &\quad d\varphi_{-t}(x)U_-(x)=e^{-\int_{-t}^0 r_-(\varphi_s(x))ds}U_-(\varphi_{-t}(x))\\
[X,U_+]=r_+U_+, &\quad d\varphi_t(x)U_+(x)=e^{-\int_0^t r_+(\varphi_s(x))ds}U_+(\varphi_t(x)).
\end{aligned}\tag{3.1}$$

We consider two smooth non-vanishing vector fields H,V such that $TM=\mathbb{R}X\oplus\mathbb{R}H\oplus\mathbb{R}V$ and so that the induced projection on H parallel to $\mathbb{R}X\oplus\mathbb{R}V$

$$\pi_H : TM\to \mathbb{R}H \tag{3.2}$$

is an isomorphism $\pi_H : E_u\to\mathbb{R}H$. For example we can consider H to be a smooth approximation of U_- and V be a smooth approximation of U_+, satisfying

$$\|U_- - H\|_{C^0}+\|U_+-V\|_{C^\gamma}<\varepsilon,\quad \|\mathcal{L}_XH-\mathcal{L}_XU_-\|_{C^0}+\|\mathcal{L}_XV-\mathcal{L}_XU_+\|_{C^0}<\varepsilon \tag{3.3}$$

for some small $\varepsilon>0$. Such approximation can be done using a partition of unity and in local charts by convolutions, the first coordinate being chosen so that $\partial_{x_1}=X$ in that chart (as in the proof of Lemma 2.19). Up to multiplying U_- by a positive C^γ function a satisfying $Xa\in C^\gamma$ and $\|a-1\|_{C^0}+\|Xa\|_{C^0}=\mathcal{O}(\varepsilon)$ (in which case it simply amounts to modify r_- by a coboundary $X\log(a)$) we can assume that

$$U_-=H+r_VV+r_XX$$

for some $r=\begin{pmatrix} r_V\\ r_X\end{pmatrix}\in C^\alpha(M;\mathbb{R}^2)$. The functions r_X, r_V characterize the regularity of the unstable foliation.

Lemma 3.1. *The function r solves a Riccati type equation*

$$0=-Xr(x)+B(x)r(x)+(Q(x)r(x))r(x)+C(x) \tag{3.4}$$

where B,Q,C are smooth matrices and for each $\varepsilon>0$, one can choose H,V so that

$$|C(x)|+|Q(x)|<\varepsilon,\qquad \left|B(x)-\begin{pmatrix} -r_+-r_- & 0\\ 0 & -r_-\end{pmatrix}\right|<\varepsilon.$$

and $Q=(Q_0\ 0)$ for $Q_0\in C^\infty(M)$. If $E_u\oplus E_s$ is smooth, for example in the contact case when $E_u\oplus E_s=\mathrm{Ker}(\alpha)$, then we can choose $H,V\in E_u\oplus E_s$, and $r_X=0$ and r_V solves

$$0=-Xr_V(x)+b(x)r_V(x)+q(x)r_V(x)^2+c(x). \tag{3.5}$$

with $|c(x)|+|q(x)|<\varepsilon$ and $|b(x)+r_+(x)+r_-(x)|<\varepsilon$

Proof. Let us write the matrix A^t of $d\varphi_t$ in the basis (H,V,X) under the form

$$A^t=\begin{pmatrix} A^t_{11} & A^t_{12} & 0\\ A^t_{21} & A^t_{22} & 0\\ A^t_{31} & A^t_{32} & 1\end{pmatrix}.$$

The invariance $d\varphi_tU_-\in\mathbb{R}U_-$ implies that $U_-^t(x):=d\varphi_t(x)U_-(x)$ can be written under the form

$$U_-^t(x)=a_t(x)\Big(H(\varphi_t(x))+r_V(\varphi_t(x))V(\varphi_t(x))+r_X(\varphi_t(x))X(\varphi_t(x))\Big)$$

for some non vanishing function $a_t(x)$. Using the matrix representation A^t of $d\varphi_t$, it is direct to see that

$$\begin{pmatrix} r_V(\varphi_t(x)) \\ r_X(\varphi_t(x)) \end{pmatrix} = \frac{1}{A^t_{11}(x) + A^t_{12}(x) r_V(x)} \begin{pmatrix} A^t_{21}(x) + A^t_{22}(x) r_V(x) \\ A^t_{31}(x) + A^t_{32}(x) r_V(x) + r_X(x) \end{pmatrix}.$$

We differentiate this equation at $t = 0$ and use that $d\varphi_0 = \mathrm{Id}$, we then get the scalar Riccati type equation

$$0 = -Xr(x) + B(x)r(x) + (Q(x)r(x))r(x) + C(x)$$

with

$$C(x) = \begin{pmatrix} \dot{A}_{21}(x) \\ \dot{A}_{31}(x) \end{pmatrix}, \quad B(x) = \begin{pmatrix} \dot{A}_{22}(x) - \dot{A}_{11}(x) & 0 \\ \dot{A}_{32}(x) & -\dot{A}_{11}(x) \end{pmatrix},$$
$$Q(x) = (-\dot{A}_{12}(x)\ 0)$$

and here we have set $\dot{A}_{ij}(x) := \partial_t A^t_{ij}(x)|_{t=0}$. In the case where $E_u \oplus E_s$ is a smooth bundle, we can choose $H, V \in E_u \oplus E_s$ and then we have $r_X = A_{31} = A_{32} = 0$, which gives the equation on r_V

$$0 = -Xr_V(x) + (\dot{A}_{22}(x) - \dot{A}_{11}(x))r_V(x) - \dot{A}_{12}(x)r_V(x)^2 + \dot{A}_{21}(x).$$

We notice that by taking H a smooth approximation (at scale $\varepsilon > 0$) of U_- and V a smooth approximation of U_+, we have as $\varepsilon \to 0$

$$A^t_{11}(x) = e^{\int_0^t r_-(\varphi_s(x))ds} + o(1), \quad A^t_{22}(x) = e^{-\int_0^t r_+(\varphi_s(x))ds} + o(1)$$
$$A^t_{12}(x) = o(1), \quad A^t_{21}(x) = o(1), \quad A^t_{31}(x) = o(1)$$

where the remainders $o(1)$ are uniform in (x, t) for small t, as well as the t-derivative. This implies, using (3.3), that with such a choice,

$$\dot{A}_{11}(x) = r_-(x) + o(1), \quad \dot{A}_{22}(x) = -r_+(x) + o(1),$$
$$\dot{A}_{12}(x) = o(1), \quad \dot{A}_{21}(x) = o(1), \quad \dot{A}_{31}(x) = o(1)$$

and that shows the desired result. □

Remark 3. In the geodesic flow case, it is known that the bundles E_u and E_s are trivial. Then taking for H the intersection of the horizontal bundle and of the kernel of the Liouville form, and for V the vertical bundle, the equation becomes the well-known Ricatti equation

$$Xr + r^2 + K = 0$$

where K is the lift of the Gauss curvature to the unit tangent bundle [Pat].

Define the subbundles E_0^*, E_u^* and E_s^* of T^*M by

$$E_u^*(E_u \oplus \mathbb{R}X) = 0, \quad E_s^*(E_s \oplus \mathbb{R}X) = 0, \quad E_0^*(E_u \oplus E_s) = 0.$$

Then we have the following result on the regularity of E_u (the same obviously holds for E_s by taking the flow in reverse time).

Theorem 2. *Let X be a smooth vector field generating an Anosov flow φ_t on a 3-dimensional closed manifold. Let $r \in C^\gamma(M, \mathbb{R}^2)$ be a solution of the Riccati equation (3.4) for $\gamma > 0$ and U_- be the unstable vector field of (3.1).*
1) Then $\mathrm{WF}(r) \subset E_u^*$ and $\mathrm{WF}(U_-) \subset E_u^*$.
2) Assume that X is volume preserving. Then $r \in H^{1-\delta}(M), U_- \in H^{1-\delta}(M)$ for any $\delta > 0$. If X is a contact flow, then $U_- \in H^{2-\delta}(M)$ for all $\delta > 0$.
3) Assume that X is volume preserving. If $r \in H^{2+\delta}(M)$, or equivalently $U_- \in H^{2+\delta}(M)$, for some $\delta > 0$, then $r \in C^\infty$ and $U_- \in C^\infty$.
4) In general, let $s > 0$ such that there is $T > 0$ large so that uniformly on M

$$(s - \frac{3}{2}) \int_0^T r_+ \circ \varphi_t \, dt > \tfrac{1}{2} \int_0^T r_- \circ \varphi_t \, dt.$$

If $U_- \in H^s(M)$, then $U_- \in C^\infty(M)$.

If the vector field is in $C^\alpha(M)$ for some $\alpha > 1$, then the same properties hold but remplacing WF by WF_{H^α} in 1), $U_- \in H^{2-\delta}(M)$ by $U_- \in H^{\min(\alpha,2)-\delta}(M)$ in 2), and $U_- \in C^\infty$ by $U_- \in H^{\alpha-\delta}(M)$ for all $\delta > 0$ in 3) and 4).

Remark. The results in 3) are weaker than what is proved in [HK90, Has92] where Hölder (in fact Zygmund) regularity $C^{1-\delta}$ and $C^{2-\delta}$ are shown to hold. The rigidity results 4) and 5) are in Sobolev norms, thus are not contained in those references (recall that $H^{2+\delta}(M)$ is only included in $C^{3/2}(M)$ for $\delta \to 0$).

Proof. First, as $r \in C^\gamma$, we also have $r \in H^s(M)$ for any $0 < \gamma' < \gamma$. Using Proposition 2.17, the equation (3.4) can then be paralinearized and rewritten as follows (with $r = (r_V \; r_X)^T$)

$$-iXr + iBr - 2iQ_0 T_{r_V} r_V = -iQ_0 R(r_V, r_V) + iC \tag{3.6}$$

where $T_{r_V} \in {}^\gamma\tilde{\Psi}^0_{1,1}(M)$ is the paradifferential operator of r_V and $R(r_V, r_V) \in H^{2\gamma'}(M)$, while $B, C, Q_0 \in C^\infty$. Note that the right hand side then belongs to $H^{2\gamma'}(M)$, thus more regular in Sobolev scale than r. We denote also by T_r the ${}^\gamma\tilde{\Psi}^0_{1,1}(M)$ valued matrix

$$T_r := \begin{pmatrix} Q_0 T_{r_V} & 0 \\ 0 & 0 \end{pmatrix} \text{ with principal symbol } \sigma(T_r) = \begin{pmatrix} Q_0 r_V & 0 \\ 0 & 0 \end{pmatrix} + o(1) \text{ as } |\xi| \to \infty.$$

The operator $P := -iX + iB - 2iT_r \in \mathrm{Diff}^1(M) + {}^\gamma\tilde{\Psi}^0_{1,1}(M) \subset {}^{1+\gamma}\tilde{\Psi}^1_{1,1}(M)$ has principal symbol given by $p_1(x,\xi) := \sigma(P)(x,\xi) = \xi(X)$. We can use the ellipticity estimates of Corollary 2.11: this gives that for each $Z \in \Psi^0(M)$ with $\mathrm{WF}(Z) \cap \{\xi(X) = 0\} = \emptyset$, $Zr \in H^{2\gamma'+1}(M)$, that is $\mathrm{WF}_{H^{2\gamma'+1}}(r) \subset \{\xi(X) = 0\}$. We fix a smooth measure μ, for example using a Riemannian metric, this induces an L^2 scalar product. The principal symbol of the imaginary part of P is

$$\sigma(\mathrm{Im}\, P) = \frac{B + B^*}{2} - 2\sigma(T_r) + \frac{\mathrm{div}(X)}{2}$$

where $\mathrm{div}(X) = \mathcal{L}_X\mu/\mu$ and we have used the measure $d\mu$ to define L^2-adjoints. By Lemma 3.1 we obtain

$$\sigma(\mathrm{Im}\, P) = \begin{pmatrix} -r_+ - r_- & 0 \\ 0 & -r_- \end{pmatrix} + \frac{\mathrm{div}(X)}{2} + \mathcal{O}(\varepsilon) + o(1) \text{ as } |\xi| \to \infty. \tag{3.7}$$

Notice that in the case of a contact flow we can choose $r_- = r_+$, $\mathrm{div}(X) = 0$ and $r_X = 0$ so that we get a scalar equation and $\sigma(\mathrm{Im}\, P) = -2r_- + \mathcal{O}(\varepsilon) + o(1)$ as $|\xi| \to \infty$. We take the canonical flat connection on the trivial $\mathbb{R}^2$-bundle over M and use the notations of Lemma 2.22, in particular $\Phi_t = e^{tH_{p_1}}$ is the Hamilton flow of p_1, i.e. the symplectic lift of φ_t in our case. Moreover $L := E_s^* \cap \partial\overline{T}^*M$ is a source for Φ_t since φ_T is Anosov. By (3.1), on E_s^* one has

$$\int_0^T \frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle} \circ \Phi_t\, dt = \log\frac{\langle\xi \circ \Phi_T\rangle}{\log\langle\xi\rangle} = \log\frac{\langle\xi \circ d\varphi_T^{-1}\rangle}{\log\langle\xi\rangle} = -\int_0^T r_- \circ \varphi_t\, dt.$$

Notice also that for $T > 0$ large

$$\int_0^T \mathrm{div}(X) \circ \varphi_t\, dt = \int_0^T (r_- - r_+) \circ \varphi_t\, dt + o(T).$$

For $s \in \mathbb{R}$, one has for $T > 0$ that on L

$$\int_0^T \left(\sigma(\mathrm{Im}\, P) + s\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}\right) \circ \Phi_t dt =$$

$$\begin{pmatrix} -\int_0^T(\frac{3}{2}r_+ + (\frac{1}{2}+s)r_-) \circ \varphi_t\, dt & 0 \\ 0 & -\int_0^T((\frac{1}{2}+s)r_- + \frac{1}{2}r_+) \circ \varphi_t\, dt \end{pmatrix} + \mathcal{O}(\varepsilon T).$$

We see that for all $s > -1/2$ this term is negative for $\varepsilon > 0$ chosen small enough. Thus we can apply Proposition 2.22 to deduce that there is $A \in \Psi^0(M)$ elliptic near E_s^* such that $Ar \in H^{2\gamma'}(M)$ for all $\gamma' < \gamma$; we thus get $\mathrm{WF}_{H^{2\gamma'}}(r) \cap \mathrm{ell}(A) = \emptyset$ for all $\gamma' < \gamma$. Since, by the Anosov property of the flow, each $(x,\xi) \in T^*M \setminus E_u^*$ is such that there is T so that $\Phi_{-T}(x,\xi) \in \mathrm{ell}(A) \cup \{\xi(X) \neq 0\}$, we can use the propagation of singularity of Proposition 2.20 to deduce that for each $A' \in \Psi^0(M)$ with $\mathrm{WF}(A') \cap E_u^* = \emptyset$, then $A'r \in H^{2\gamma'}(M)$, i.e. $\mathrm{WF}_{H^{2\gamma'}}(r) \subset E_u^*$, $\forall \gamma' < \gamma$.

Now take $S \in \Psi^0(M)$ such that $\mathrm{WF}(S-1) \subset O_u$ and $\mathrm{WF}(S) \cap E_u^* = \emptyset$, where O_u is an arbitrarily small conic neighborhood of E_u^*. Applying S to (3.6), we get

$$PSr = -i(SQ_0R(r_V, r_V) + SC) + [P, S-1]r.$$

By applying Lemma 2.18 with $a = b = r_V$, $\varepsilon = \gamma$, $\alpha = \beta = \gamma'$ and $\delta = \gamma'$, using $\mathrm{WF}_{H^{2\gamma'}}(r) \subset E_u^*$ we see that $\mathrm{WF}_{H^{3\gamma'}}(Q_0SR(r_V, r_V)) \subset E_u^*$ and we also have $\mathrm{WF}_{H^{3\gamma'}}([P, S-1]r) \subset O_u$ since $\mathrm{WF}(S-1) \subset O_u$. We can then apply the same argument as above where now the right hand side is in $H^{3\gamma'}(M)$ microlocally outside O_u, and we obtain that $\mathrm{WF}_{H^{4\gamma'}}(Sr) \subset O_u$. Since O_u was chosen arbitrarily small, we conclude that $\mathrm{WF}_{H^{4\gamma'}}(r) \subset E_u^*$. Then we bootstrap this argument and obtain that

$$\mathrm{WF}(r) \subset E_u^*, \quad \mathrm{WF}(U_-) \subset E_u^*.$$

Next, we apply the sink radial estimate of Proposition 2.23. We already know that $Br \in H^N(M)$ for all $N \geq 0$ if $A \in \Psi^0(M)$ has $\mathrm{WF}(A) \cap E_u^* = \emptyset$. As above, we have on $L' = E_u^* \cap \partial \overline{T}^* M$. For $s \in \mathbb{R}$, one has for $T > 0$ that on L'

$$\int_0^T \left(\sigma(\mathrm{Im\,P}) + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle}\right) \circ \Phi_t \, dt =$$
$$\begin{pmatrix} -\int_0^T (r_+(\frac{3}{2}-s) + \frac{1}{2}r_-) \circ \varphi_t \, dt & 0 \\ 0 & -\int_0^T (\frac{1}{2}r_- + (\frac{1}{2}-s)r_+) \circ \varphi_t \, dt \end{pmatrix} + \mathcal{O}(\varepsilon T).$$

This is negative for T large (and $\varepsilon > 0$ chosen small) if, uniformly on M,

$$s \int_0^T r_+ \circ \varphi_t \, dt < \frac{1}{2} \int_0^T (r_+ + r_-) \circ \varphi_t \, dt. \tag{3.8}$$

For such s we have that $r \in H^s(M)$ if $r \in H^{s-\gamma'}(M)$ for some $\gamma' < \gamma$. In the volume preserving case, the condition (3.8) holds when $s < 1$. When the flow is contact $r_X = 0$ and $r_+ = r_-$ so that one has a single equation and the term above becomes negative when $s < 2$, which shows that $r \in H^{2-\delta}(M)$ for all $\delta > 0$.

To prove the rigidity result, we apply the source estimate to the flow in reverse time. Let now $P_- := -P = iX - i - 2iT_r$ and we analyze the PDE $P_- r = iQ_0 R(r_V, r_V) - iC$. The set $L' = E_u^* \cap \partial \overline{T}^* M$ becomes a source for the Hamilton vector field $H_{-p_1} = -H_{p_1}$ where $-p_1 = \sigma(P_-) = -\xi(X)$, whose flow is given by Φ_{-t}. For $s \in \mathbb{R}$, let us analyze the negativity condition on L' of

$$\int_0^T -\left(\sigma(\mathrm{Im}\,P) + s\frac{H_{p_1}\langle\xi\rangle}{\langle\xi\rangle}\right) \circ \Phi_{-t} dt =$$
$$\begin{pmatrix} \int_{-T}^0 ((\frac{3}{2}-s)r_+ + \frac{1}{2}r_-) \circ \varphi_t \, dt & 0 \\ 0 & \int_{-T}^0 (\frac{1}{2}-s)r_+ + \frac{1}{2}r_-) \circ \varphi_t \, dt \end{pmatrix} + \mathcal{O}(\varepsilon T).$$

Taking $\varepsilon > 0$ arbitrarily small and T large enough, we see that if uniformly on M

$$(s - \frac{3}{2}) \int_0^T r_+ \circ \varphi_t \, dt > \frac{1}{2} \int_0^T r_- \circ \varphi_t \, dt \tag{3.9}$$

then we can apply Proposition 2.22 to deduce that if $r \in H^s$, then in fact r has the regularity of the right hand side $iQ_0 R(r_V, r_V) - iC \in H^{s+\gamma'}(M)$. Bootstrapping the argument we get that $r \in C^\infty(M)$. In particular, if X preserves a smooth measure, the condition is always satisfied if $s > 2$ since $\int_0^T r_- \circ \varphi_t dt = \int_0^T r_+ \circ \varphi_t dt + o(T)$ for large T.

Let us briefly discuss the case where the vector field $X \in C^\alpha$ for $\alpha > 1$ and assume $\gamma < \alpha$. Using Lemma 2.14, we can write $-iX = T_p + \mathrm{Op}(p^\flat)$ where $p(x,\xi) = \xi(X) \in C^\alpha S^1(M)$, $p^\flat \in C^{\alpha-1} S^{1-\alpha}_{1,1}(M)$ and $T_p \in {}^\alpha\tilde{S}^1_{1,1}(M)$. By (2.4), $\mathrm{Op}(p^\flat) : H^{s+1-\alpha}(M) \to H^s(M)$ for all $s \in (0,\alpha)$. Let $\beta := \min(\alpha, \gamma+1)$. The equation (3.6) can be rewritten as

$$T_p r + iBr - 2iQ_0 T_{r_V} r_V = -iQ_0 R(r_V, r_V) + iC - \mathrm{Op}(p^\flat) r \in H^{\min(2\gamma', \gamma'+\alpha-1)}(M)$$

for all $\gamma' < \min(\gamma, 1)$. Using the ellipticity result from Corollary 2.11, r is microlocally in $H^{2\gamma'}(M)$ outside $\xi(X) = 0$ (we have used that $2\gamma' < \gamma' + \alpha$). We can then apply Propositions 2.22, 2.20 as above, we obtain that r is microlocally $H^{1-\delta}(M)$ outside E_u^* for all $\delta > 0$ in the volume preserving case. In the contact case that gives that u is microlocally in $H^{\min(\alpha,2)-\delta}(M)$ for all $\delta > 0$. Using Proposition 2.23 as above, we also directly obtain that if $u \in H^s(M)$ for s satisfying (3.9), then $r \in H^{\alpha-\delta}(M)$ for all $\delta > 0$. If the flow is volume preserving and C^α for $\alpha > 2$, this shows that $r \in H^s(M)$ with $s > 2$ implies $s \in H^{\alpha-\delta}(M)$ for all $\delta > 0$. □

3.2. **Higher dimension.** Let M be a smooth compact manifold of dimension d and let X be a smooth vector field generating an Anosov flow φ_t. We denote by E_u and E_s the unstable and stable bundles as above. They are C^γ Hölder continuous for some $\gamma > 0$, [HPS70]. We denote by $n_u = \dim E_u$ and $n_s = \dim E_s$ so that $d = 1 + n_u + n_s$.

We can construct $H \subset TM$ a smooth subbundle of dimension n_u which can be chosen arbitrarily close to E_u, viewed as points in the Grassmanian $\mathcal{G}_{n_u}(M)$ of n_u dimensional subpaces in TM. Similarly let V be a smooth approximation of the weak unstable bundle $E_{s,0} := E_s \oplus \mathbb{R}X$. Due to our choices, there are smooth projections

$$\pi_H : TM \to H, \quad \pi_V : TM \to V$$

induced by the decomposition $TM = H \oplus V$ and π_H is an isomorphism when restricted to a neighborhood $W_H \subset \mathcal{G}_{n_u}(M)$ corresponding to vector subbundles close to H (for example contained in an unstable cone of E_u). In particular

$$\pi_H(E_u) : E_u \to H$$

is a C^γ Hölder isomorphism of bundles, with a C^γ Hölder inverse π_H^{-1}. We have

Lemma 3.2. *There is a one-to-one smooth correspondance between C^γ- subbundles of dimension n_u near E_u and the space $C^\gamma(M; \mathcal{L}(H, V))$ of C^γ linear maps from H to V, given by*

$$\Psi : E \in W_H \mapsto \pi_V \circ \pi_H(E)^{-1}$$

Proof. The map is smooth, and its inverse is given by the map which to $\hat{U} \in C^\gamma(M; \mathcal{L}(H, V))$ assigns the bundle given by the graph of $\hat{U}$, i.e. the image of the linear map

$$H \to TM = H \oplus V, \quad h \mapsto h + \hat{U}h.$$

The regularity of $\Psi(E)$ is easily determined by considering a local basis of smooth sections of H, V and a C^α sections of E, and then writing the $\mathcal{L}(H, V)$ map $\pi_V \circ \pi_H(E)^{-1}$ as a matrix in these basis: if $(T_j)_{j=1,\dots,n_u}$ is a basis of H, $(T_j)_{j=n_u+1,\dots,d}$ is a basis of V, and $(U_j)_{j=1,\dots,n_u}$ is a C^α basis of E, then the matrix representing the map $\Psi(E)$ in the basis $(T_j)_j$ is given by

$$\hat{U} = U^V(U^H)^{-1}$$

where $U_j = \sum_{k=1}^{n_u} U^H_{kj} T_k + \sum_{k=n_u+1}^{d} U^V_{kj} T_k$. □

The flow $d\varphi_t : TM \to TM$ also acts on the Grassmannian $\mathcal{G}_{n_u}(M)$ by simply writing
$$\Phi_t : \mathcal{G}_{n_u}(M) \to \mathcal{G}_{n_u}(M), \quad \Phi_t(E) = d\varphi_t(E).$$
We would like to describe this action in terms of the linear maps described in Lemma 3.2.

Lemma 3.3. *The action of $\Psi\Phi_t\Psi^{-1}$ on $\mathcal{L}(H,V)$ can be described as follows: let*
$$A(t) = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}$$
be the block decomposition of $d\varphi_t : H \oplus V \to H \oplus V$, then one has
$$\Psi\Phi_t\Psi^{-1}\hat{U} = (A_3(t) + A_4(t)\hat{U})(A_1(t) + A_2(t)\hat{U})^{-1} \tag{3.10}$$

Proof. Let us use the bases and notations of the proof of Lemma 3.2. For small t, we get for each $x \in M$
$$A(t)U_j(x) = \sum_{i=1}^{d} U_{kj}(x)A_{ik}(t;x)T_i(\varphi_t(x))$$
where $A_{ik}(t;x)$ are the matrix components of $A(t)$ at the point x in the basis $(T_j)_j$, and U_{jk} are the matrix components of (U^H, U^V) in $(T_j)_j$, i.e. for $a_k \in \mathbb{R}$
$$A(t)(\sum_{k=1}^{d} a_k T_k(x)) = \sum_{i,k=1}^{d} A_{ik}(t;x)a_k T_i(\varphi_t(x)).$$
This implies that
$$\begin{aligned}\Psi\Phi_t\Psi^{-1}\hat{U} =&(A_3(t)U^H + A_4(t)U^V)(A_1(t)U^H + A_2(t)U^V)^{-1}\\ =&(A_3(t) + A_4(t)\hat{U})(A_1(t) + A_2(t)\hat{U})^{-1}.\end{aligned}$$
This concludes the proof. $\square$

As a corollary, we shall describe the infinitesimal action as a Riccati operator. First, to make invariant sense of derivatives on $\mathcal{L}(H,V) = V \otimes H^*$, we fix a Riemannian metric g so on TM (here we use H^* for the dual of H, not the annihilator as we did for E_u^*, E_s^* before). We recall that we can differentiate a section $v \otimes h^* \in V \otimes H^*$ using the Lie derivative by viewing it as an element in $TM \otimes T^*M$:
$$\mathcal{L}_X(v \otimes h^*)_x = \partial_t\Big(d\varphi_{-t}(\varphi_t(x))v_{\varphi_t(x)} \otimes d\varphi_t(x)^\top h^*_{\varphi_t(x)}\Big)\Big|_{t=0} \in TM \otimes T^*M.$$

As in dimension 3, using a partition of unity, local charts and regularizations through convolutions with $\partial_{x_1} = X$ in those charts, for each $\varepsilon > 0$ we can choose H, V (equivalently π_H, π_V) and g so that (using a fixed background metric on TM for defining the norms)
$$\begin{gathered}\|\pi_H - \pi_{E_u}\|_{C^0} + \|\pi_V - \pi_{E_{s0}}\|_{C^0} < \varepsilon, \quad \|g\|_{C^0} + \|\mathcal{L}_X g\|_{C^0} \le 1\\ \|\mathcal{L}_X(\pi_H - \pi_{E_u})\|_{C^0} + \|\mathcal{L}_X(\pi_V - \pi_{E_{s0}})\|_{C^0} < \varepsilon.\end{gathered} \tag{3.11}$$
where $\pi_{E_u} : TM \to E_u$ and $\pi_{E_{s0}} : TM \to E_s \oplus \mathbb{R}X$ are the C^γ projections induced by the decomposition $TM = E_u \oplus (E_s \oplus \mathbb{R}X)$. We can also assume that $|g(X,v)| < \varepsilon\|v\|_g$ for all

$v \in E_s$, for some arbitrarily small $\varepsilon > 0$ and that g is close enough (in Hölder norm) to some adapted metric for X so that

$$\begin{aligned} &\forall t \geq 0, \quad (1-\varepsilon)e^{(-\nu_u^{\max}-\varepsilon)t} \leq \|d\varphi_{-t}|_{E_u}\|_g \leq (1+\varepsilon)e^{(-\nu_u^{\min}+\varepsilon)t}, \\ &\forall t \geq 0, \quad (1-\varepsilon)e^{(-\nu_s^{\max}-\varepsilon)t} \leq \|d\varphi_{t}|_{E_s}\|_g \leq (1+\varepsilon)e^{(-\nu_s^{\min}+\varepsilon)t} \end{aligned} \tag{3.12}$$

where $\nu_u^{\min}, \nu_u^{\max}$ are the minimal expansion rates of the flow on E_u and similarly for E_s.

Corollary 3.4. *Let $\hat{U} \in C^\alpha(M; \mathcal{L}(H,V))$ be the representation of a vector subbundle of dimension n_u close to E_u in $\mathcal{G}_{n_u}(M)$. This subbundle is invariant by $d\varphi_t$ if and only if*

$$\mathcal{L}_X\hat{U} + \hat{U}\dot{A}_2\hat{U} + \hat{U}\dot{A}_1 - \dot{A}_4\hat{U} - \dot{A}_3 = 0 \tag{3.13}$$

where

$$\begin{aligned} \dot{A}_3(x) &= \partial_t(d\varphi_{-t}(\varphi_t(x))\pi_V d\varphi_t(x)\pi_H)|_{t=0}, \quad \dot{A}_4(x) = \partial_t(d\varphi_{-t}(\varphi_t(x))\pi_V d\varphi_t(x)\pi_V)|_{t=0} \\ \dot{A}_1(x) &= \partial_t(d\varphi_t(x)^{-1}\pi_H d\varphi_t(x)\pi_H|_{t=0}, \quad \dot{A}_2(x) = \partial_t(d\varphi_t(x)^{-1}\pi_H d\varphi_t(x)\pi_V)|_{t=0}. \end{aligned}$$

Moreover for each $\varepsilon > 0$, one can choose H, V arbitrarily close to E_u and $E_s \oplus \mathbb{R}X$ so that uniformly on M

$$\sum_{j=1}^{4} \|\dot{A}_j\|_{C^0} \leq \varepsilon.$$

Proof. Let us define $\hat{U}^t := \Psi\Phi_t\Psi^{-1}\hat{U}$. The subbundle $\hat{U}$ is invariant if and only if $\hat{U}^t(\varphi_{-t}(x)) = \hat{U}(x)$ for all $x \in M$ and all $t \in \mathbb{R}$. We have

$$\begin{aligned} \mathcal{L}_X\hat{U} =& \partial_t\Big(d\varphi_{-t}(\varphi_t(\cdot))\hat{U}(\varphi_t(\cdot))d\varphi_t(\cdot)\Big)\Big|_{t=0} = \partial_t\Big((\tilde{A}_3(t) + \tilde{A}_4(t)\hat{U})(\tilde{A}_1(t) + \tilde{A}_2(t)\hat{U})^{-1}\Big)\Big|_{t=0} \\ =& -\hat{U}\dot{A}_2\hat{U} - \hat{U}\dot{A}_1 + \dot{A}_4\hat{U} + \dot{A}_3 \end{aligned}$$

with $\dot{A}_j := \partial_t\tilde{A}_j(t;x)|_{t=0}$ and

$$\begin{aligned} \tilde{A}_3(t;x) &= d\varphi_{-t}(\varphi_t(x))\pi_V d\varphi_t(x)\pi_H, \quad \tilde{A}_4(t;x) = d\varphi_{-t}(\varphi_t(x))\pi_V d\varphi_t(x)\pi_V \\ \tilde{A}_1(t;x) &= d\varphi_t(x)^{-1}\pi_H d\varphi_t(x)\pi_H, \quad \tilde{A}_2(t;x) = d\varphi_t(x)^{-1}\pi_H d\varphi_t(x)\pi_V. \end{aligned}$$

Fianlly, we choose $H \oplus V$ to satisfy (3.11), and we obtain the desired properties $\|\dot{A}_j\|_{C^0} = \mathcal{O}(\varepsilon)$ for all j. ◻

Theorem 3. *Let X be a smooth vector field generating an Anosov flow φ_t preserving a smooth measure on M. Let $\hat{U} \in C^\gamma(M; \mathcal{L}(V \otimes H^*))$ be the section parametrizing the bundle E_u and $\nu_u^{\min}, \nu_u^{\max}, \nu_s^{\min}, \nu_s^{\max}$ the minimal/maximal expansion rates of the flow, see (3.12).*
1) Then its wavefront set satisfies $\mathrm{WF}(\hat{U}) \subset E_u^*$ and $\hat{U}$ has Sobolev regularity $H^s(M)$ near E_u^* for all $s < (\nu_u^{\min} + \nu_s^{\min})/\nu_s^{\max}$.
2) If $\hat{U} \in H^s(M)$, and thus $E_u \in H^s$, for $s > (\nu_u^{\max} + \nu_s^{\max})/\nu_s^{\min}$, then $\hat{U} \in C^\infty$ and $E_u \in C^\infty$.

In the case of a C^α vector field for $\alpha > 1$, the results also holds just as in Theorem 2.

Proof. We apply the same proof as Theorem 2 in dimension 3 and do not repeat the argument but only the main necessary eventual negativity estimate to apply Propositions 2.22 and 2.23. We can rewrite the equation (3.13) using the paraproduct under the form

$$-i\mathcal{L}_X\hat{U}-i(\hat{U}\dot{A}_1-\dot{A}_4\hat{U})-i(T_{\hat{U}}\dot{A}_2\hat{U}+(T_{\hat{U}^\top}\dot{A}_2^\top\hat{U}^\top)^\top)=iR(\hat{U},\hat{U})-i\dot{A}_3 \tag{3.14}$$

where $T_{\hat{U}}\in{}^\gamma\tilde{\Psi}^0_{1,1}(M;\mathcal{L}(H,V))$ is the paradifferential operator of $\hat{U}$ and $R(\hat{U},\hat{U})\in H^{2\gamma'}(M)$ for all $\gamma'<\gamma$, while $\dot{A}_j\in C^\infty$, and we use $^\top$ to denote the transpose of the linear maps. Note that the right hand side then belongs to $H^{2\gamma}(M)$, thus more regular in Sobolev scale than $\hat{U}$. We write P to be the operator appearing on the left hand side (applied to $\hat{U}$), its principal symbol is $p_1=\xi(X)$ as in dimension 3. Let us compute the subprincipal term $\sigma(\operatorname{Im}P)$: since $\|\dot{A}_j\|_{C^0}=\mathcal{O}(\varepsilon)$ for all $j=1,\dots,4$, we have (using that X is volume preserving)

$$\operatorname{Im}P=-\frac{1}{2}\sigma(\mathcal{L}_X^*+\mathcal{L}_X)+\mathcal{O}(\varepsilon)=\frac{1}{2}\mathcal{L}_X\mathbf{g}+\mathcal{O}(\varepsilon)$$

where we have used, for $\mathbf{g}$ the metric induced by g on $TM\otimes T^*M$, that

$$X(\mathbf{g}(\hat{U}_1,\hat{U}_2))=\mathcal{L}_X\mathbf{g}(\hat{U}_1,\hat{U}_2)+\mathbf{g}(\mathcal{L}_X\hat{U}_1,\hat{U}_2)+\mathbf{g}(\hat{U}_1,\mathcal{L}_X\hat{U}_2).$$

One has for $v\in V,h^*\in H^*$ with $\|v\|_g\le 1$ and $\|h^*\|_g\le 1$

$$\begin{aligned}\mathcal{L}_X\mathbf{g}(v\otimes h^*,v\otimes h^*)=&\partial_t(\|d\varphi_tv\|^2_{g_{\varphi_t}}\times\|d\varphi_t^{-\top}h^*\|^2_{g_{\varphi_t}})|_{t=0}\\ =&2\partial_t(\|d\varphi_t\pi_{E_{s0}}v\|_{g_{\varphi_t}})|_{t=0}+2\partial_t(\|d\varphi_t^{-\top}\pi_{E_u}^\top h^*\|_{g_{\varphi_t}})|_{t=0}+\mathcal{O}(\varepsilon).\end{aligned}$$

with the notation $A^{-\top}=(A^\top)^{-1}$. Consider the quadratic forms

$$\mathcal{R}_u(h^*,h^*)=2\partial_t(\|d\varphi_t^{-\top}\pi_{E_u}^\top h^*\|_{g_{\varphi_t}})|_{t=0}\|h^*\|_g,\quad \mathcal{R}_{s0}(v,v)=2\partial_t(\|d\varphi_t\pi_{E_{s0}}v\|_{g_{\varphi_t}})|_{t=0}\|v\|_g$$

on respectively T^*M and TM. By (3.12), we obtain that (recall $\partial_t\|d\varphi_tX\|_g=0$)

$$\mathcal{R}_u(h^*,h^*)\|v\|_g^2+\mathcal{R}_{s0}(v,v)\|h^*\|_g^2<(-2\nu_u^{\min}-2\nu_s^{\min}+\mathcal{O}(\varepsilon))\|v\otimes h^*\|_g^2$$

for $\varepsilon>0$ small. Note also, using again (3.12), that for $\xi\in E_s^*$ and $p_1=\xi(X)$

$$H_{p_1}\log\langle\xi\rangle=\partial_t(\log\langle\xi\circ\Phi_t\rangle)|_{t=0}\le-\nu_u^{\min}+\varepsilon$$

The negativity of the subprincipal term needed to apply Proposition 2.22 at the source $E_s^*\cap\partial\overline{T}^*M$ is satisfied for all $s>0$:

$$\begin{aligned}\frac{1}{2}(\mathcal{R}_u(h^*,h^*)\|v\|_g^2+\mathcal{R}_{s0}(v,v)\|h^*\|_g^2)+sH_{p_1}\log\langle\xi\rangle\|v\otimes h^*\|_g^2+\mathcal{O}(\varepsilon)\\ <-(\nu_u^{\min}+\nu_s^{\min}+s\nu_u^{\min}+\mathcal{O}(\varepsilon))\|h^*\|_g^2\|v\|_g^2\end{aligned}$$

Since $\varepsilon>0$ is arbitrarily small, the proof of Theorem 2 then shows that $\mathrm{WF}(\hat{U})\subset E_u^*$.

The regularity of $\hat{U}$ at E_u^* can be obtained by Proposition 2.23 by considering the maximal $s>0$ so that the subprincipal term on $L'=E_u^*\cap\partial\overline{T}^*M$

$$\begin{aligned}\frac{1}{2}(\mathcal{R}_u(h^*,h^*)\|v\|_g^2+\mathcal{R}_{s0}(v,v)\|h^*\|_g^2)+sH_{p_1}\log\langle\xi\rangle\|v\otimes h^*\|_g^2+\mathcal{O}(\varepsilon)\\ <-(\nu_u^{\min}+\nu_s^{\min}+s\nu_s^{\max}+\mathcal{O}(\varepsilon))\|h^*\|_g^2\|v\|_g^2\end{aligned}$$

is negative. Taking $\varepsilon > 0$ sufficiently small, we see that $s = (\nu_u^{\min} + \nu_s^{\min})/\nu_s^{\max}$ is a lower bound on the threshold.

For the rigidity, we reverse the direction of the flow so that E_u^* becomes the source and the threshold condition becomes (as in the proof of Theorem 2) on L' that

$$-\frac{1}{2}(\mathcal{R}_u(h^*,h^*)\|v\|_g^2 + \mathcal{R}_{s0}(v,v)\|h^*\|_g^2) + sH_{p_1}\log\langle\xi\rangle\|v\otimes h^*\|_g^2 + \mathcal{O}(\varepsilon)$$
$$< (\nu_u^{\max} + \nu_s^{\max} - s\nu_s^{\min} + \mathcal{O}(\varepsilon))\|h^*\|_g^2\|v\|_g^2$$

is negative, which is satisfied, if $\varepsilon > 0$ is chosen small enough, when $s > (\nu_u^{\max} + \nu_s^{\max})/\nu_s^{\min}$. In that case, the bootstrap argument can be performed and we obtain that $\hat{U}$ is smooth. □

Remark 4. As mentionned in the introduction, this method also gives directly some regularity and rigidity statements of the same kind for general Riccati equations

$$\mathcal{L}_X U + Q(x,U) = 0$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of a smooth Anosov vector field, Q is a quadratic polynomial in U (or even more generally a smooth functional) depending smoothly on x, $U \in C^r(M;\mathrm{End}(E))$ is a Hölder section of some smooth Hermitian bundle (E,g) equipped with a natural lifted action $\tilde{\varphi}_t : E_x \to E_{\varphi_t(x)}$ of the flow φ_t on E that is linear in the fibers. Indeed, using Bony's paralinearization [Bon81, Proposition 4.4., Theorème 4.5], the equation can be replaced by

$$\mathcal{L}_X U + \partial_U Q(x,U).U = R(U)$$

for some $R(u) \in C^{2r}(M;\mathrm{End}(E))$. We then see that the radial point condition to apply Proposition 2.22 near the source $L = E_s^* \cap \partial\overline{T}^* M$ can be written in terms of the condition

$$\frac{1}{2}(\mathcal{L}_X \mathbf{g} + \mathrm{div}(X)) - \frac{1}{2}(\partial_U Q(x,U) + \partial_U Q(x,U)^*)dt - s\nu_u^{\min} < 0 \tag{3.15}$$

where $\mathbf{g}$ is the metric on $\mathrm{End}(E)$ induced by a metric g on E, and the adjoint is taken with respect to the metric g. As before, $\mathcal{L}_X \mathbf{g}$ can be expressed in terms of expansion rates of the lifted flow $\tilde{\varphi}_t$ on the bundle E, more precisely as $\partial_t\|\tilde{\varphi}_t\|_{\mathbf{g}}^2|_{t=0} + \partial_t\|\tilde{\varphi}_t^{-\top}\|_{\mathbf{g}}^2|_{t=0}$. The radial sink estimate can be applied provided

$$-\frac{1}{2}(\mathcal{L}_X \mathbf{g} + \mathrm{div}(X)) + \frac{1}{2}(\partial_U Q(x,U) + \partial_U Q(x,U)^*)dt - s\nu_s^{\min} < 0. \tag{3.16}$$

If $E = \mathbb{R}$ is the trivial bundle, one can replace this condition by $\int_0^T a\circ\varphi_t dt < 0$ for large $T > 0$ where a is the quantity in (3.15) and (3.16) without the term $\mathcal{L}_X \mathbf{g}$.

4. Ruelle resonances for non-smooth potentials

The notion of Ruelle resonances for non-smooth Anosov flows has been defined by Butterley-Liverani [BL07], it was previously done for hyperbolic diffeomorphisms by Baladi-Tsujii [BT07] and Gouezel-Liverani [GL08] (including non-smooth potentials in that case). Here we give a short application of the paradifferential calculus mentionned above to the study of $P = -X + V$ where X is a smooth Anosov flow on a compact manifold M and $V \in C^r(M)$ a Hölder potential. For simplicity we will assume that there is a smooth invariant measure μ. The motivation of considering non-smooth potentials comes from the fact that the geometric potentials such as the unstable Jacobian $V = J_u := \partial_t(\log\det(d\varphi_t|_{E_u}))|_{t=0}$ are never smooth except for locally symmetric spaces. One can also consider non-smooth flows using this technique, we refer to the Appendix by Guedes Bonthonneau for the more general result.

By Faure-Sjöstrand [FS11], there is $C_0 > 0$ such that for all $u < 0 < s$, for all $\varepsilon > 0$ small, there is a smooth order function $m \in S^0(M; [u, s])$ of order 0 on T^*M such that, if H_p is the Hamilton flow of $p(x, \xi) = \xi(X)$, then

$$H_p m \geq 0, \text{on } M, \quad m(x,\xi) = s \text{ near } E_s^*, \quad m(x,\xi) = u \text{ near } E_u^*$$

with $s > 0 > u$ and a function $f : T^*M \to \mathbb{R}^+$ homogeneous of degree 1 for $|\xi| > 1$, so that if $G := m \log f$, one has for $\varepsilon > 0$ arbitrarily small,[6]

$$\begin{aligned} &H_p G \leq -\min(\nu_u^{\min}|u|, s\nu_s^{\min}) + C_0\varepsilon(|u| + s) \text{ outside a conic neighborhood of } E_0^* \\ &H_p G \leq 0 \text{ on } T^*M. \end{aligned} \tag{4.1}$$

Here $\nu_u^{\min}, \nu_s^{\min}$ are the minimal expansion rates of the flow in E_u, E_s. The anisotropic Sobolev space is defined in [FS11] by

$$\mathcal{H}^{s,u} := \mathrm{Op}(e^{-G})L^2(M). \tag{4.2}$$

Proposition 4.1. *Let X be a smooth Anosov vector field with flow φ_t preserving a smooth mesure, $V \in C^r(M)$. There is $C_0 > 0$ such that for $u < 0 < s$ so that $s + |u| < r$, all $\varepsilon > 0$, the operator $P = -X + V$ has only discrete spectrum on the Hilbert space $\mathcal{H}^{s,u}$ defined in (4.2), in the region*

$$\mathrm{Re}(\lambda) > -\min(\nu_u^{\min}|u|, s\nu_s^{\min}) + \|\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{Re}(V) \circ \varphi_t \, dt\|_{L^\infty} + C_0\varepsilon(|u| + s)$$

with meromorphic resolvent there, and $P - \lambda : \mathrm{Dom}_{\mathcal{H}^{s,u}}(P) \to \mathcal{H}^{s,u}$ is Fredholm in that region.

Proof. First, notice that, using the same argument as in Lemma 2.21, we can first conjugate $-X + V$ by e^b for some $b \in C^\infty(M)$ so that we can replace V by $V_0 \in C^\infty(M)$ such that

$$\mathrm{Re}(V_0) \leq \|\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{Re}(V) \circ \varphi_t \, dt\|_{L^\infty} + \varepsilon_0$$

for $\varepsilon_0 > 0$ arbitrarily small. For simplicity, we keep the notation V instead of V_0 in what follows. Using Lemma 2.14, write $P = P^\sharp + V^b$ where $P^\sharp = -X + T_V$ is the paradifferential operator of P: one has $T_V \in {}^r\tilde{\Psi}^0_{1,1}(M)$ and $V^\flat \in \Psi^{-r}_{1,1}(M)$ maps continuously

$$V^\flat : H^{\alpha - r}(M) \to H^\alpha, \quad \forall \alpha \in (0, r).$$

[6]The constant c in [FS11, eq 1.19] is readily checked to be what we give here, for example using an adapted metric as in the previous Section.

It then suffices to proceed exactly as in [FS11]. Consider

$$\tilde{P} := \mathrm{Op}(e^{-G})^{-1}P^\sharp \mathrm{Op}(e^{-G}) + \mathrm{Op}(e^{-G})^{-1}V^\flat \mathrm{Op}(e^{-G}) =: \tilde{P}^\sharp + \tilde{V}^\flat.$$

Since $\mathrm{Op}(e^{-G}) : L^2(M) \to H^u(M)$ and $\mathrm{Op}(e^{-G})^{-1} : H^{u+r}(M) \to H^{u+r-s}(M)$ (eg. see [FRS08, FS11]), we deduce that (recall $T_V = \mathrm{Op}(V^\sharp)$)

$$\tilde{V}^\flat : L^2(M) \to H^{u+r-s}(M) \tag{4.3}$$

continuously. In particular it is compact on L^2 if $r > s + |u|$. Next we analyze $\tilde{P}^\sharp$. The term $\mathrm{Op}(e^{-G})^{-1}P^\sharp\mathrm{Op}(e^{-G})$ is studied in [FS11] and gives[7] for all $\delta > 0$ small

$$\mathrm{Op}(e^{-G})^{-1}P^\sharp\mathrm{Op}(e^{-G}) = -X - \mathrm{Op}(H_pG) + \mathcal{O}_{\Psi^{-1+\delta}(M)}(1)$$

To analyze the term $\mathrm{Op}(e^{-G})^{-1}P^\sharp\mathrm{Op}(e^{-G})$, we can use Proposition 2.8 and the fact that $e^{-G} \in S^{|u|}_{1-\delta,\delta}(M)$ for all $\delta > 0$ to deduce that

$$\mathrm{Op}(V^\sharp)\mathrm{Op}(e^{-G}) = \mathrm{Op}(V_1) + \mathcal{O}_{\Psi^{-\infty}(M)}(1), \quad \mathrm{Op}(e^{-G})^{-1}\mathrm{Op}(V_1) = \mathrm{Op}(V_2) + \mathcal{O}_{\Psi^{-\infty}(M)}(1)$$

$$V_1(x,\xi) = \sum_{|\alpha|<\lceil r\rceil} \frac{1}{\alpha!}\partial_\xi^\alpha V^\sharp(x,\xi) D_x^\alpha e^{-G(x,\xi)} + \mathcal{O}_{S^{|u|-r}_{1,1}(M)}(1) \in {}^r\tilde{S}^{|u|}_{1,1}(M)$$

$$V_2(x,\xi) = \sum_{|\alpha|<\lceil r\rceil} \frac{1}{\alpha!}\partial_\xi^\alpha (e^{G(x,\xi)}F) D_x^\alpha V_1(x,\xi) + \mathcal{O}_{S^{|u|+s-r}_{1,1}(M)}(1) \in {}^r\tilde{S}^{|u|+r}_{1,1}(M)$$

where we wrote $\mathrm{Op}(e^{-G})^{-1} = \mathrm{Op}(e^G F) + \mathcal{O}_{\Psi^{-\infty}(M)}(1)$ for some $F \in S^0(M)$. It is readily seen that for all $\delta > 0$

$$V_2(x,\xi) = V^\sharp(x,\xi) + \mathcal{O}_{S^{-1+\delta}_{1,1}(M)}(1) + \mathcal{O}_{S^{s+|u|-r}_{1,1}(M)}(1).$$

Thus we get

$$\tilde{P}^\sharp = -X - \mathrm{Op}(H_pG) + \mathrm{Op}(V^\sharp) + \mathcal{O}_{\Psi^{-\gamma}_{1,1}(M)}(1)$$

for all $\gamma < \min(r - s - |u|, 1)$. Now we can apply the proof of [FS11, Lemma 3.3,3.4,3.5] verbatim, noting that the only needed fact is the Gårding inequality Proposition 2.12: the principal symbol of the subprincipal term $\mathrm{Im}(i(\tilde{P}^\sharp - \lambda)) = -\frac{1}{2}((\tilde{P}^\sharp)^* + \tilde{P}^\sharp) - \mathrm{Re}(\lambda)$ satisfies in $\{|\xi| > R\}$ for $R \gg 1$ large enough

$$\begin{aligned}\sigma(\mathrm{Im}(i(\tilde{P}^\sharp - \lambda))) =& H_P G + \mathrm{Re}(V) - \mathrm{Re}(\lambda) + o(1)\\ \leq& -\min(\nu_u^{\min}|u|, s\nu_s^{\min})(1-\chi_0^2) + \mathrm{Re}(V) - \mathrm{Re}(\lambda) + C_0\varepsilon(|u|+s)\end{aligned}$$

for some $\chi_0 \in S^0(M)$ supported in a conic neighborhood of E_0^*. This term is bounded above by $-C_0\varepsilon(|u|+s)/2 + C_{s,u}\chi_0^2$ for some $C_{s,u} > 0$ if $\lambda \in \Omega := \{\mathrm{Re}(\lambda) > -\min(\nu_u^{\min}|u|, s\nu_s^{\min}) + \mathrm{Re}(V) + C_0\varepsilon(|u|+s)/2\}$, thus by Proposition 2.12

$$\langle \mathrm{Im}(i(\tilde{P}^\sharp - \lambda + C_{s,u}\mathrm{Op}(\chi_0^2)))u, u\rangle_{L^2} \leq -\frac{C\varepsilon}{2}\|u\|^2_{L^2} + C'\|u\|^2_{H^{-r/4+\varepsilon}}.$$

[7] We use that $X^* = -X$ on $L^2(M, d\mu)$ since X preserves a smooth measure.

for some constant $C' > 0$. Combining with (4.3), we obtain an estimate for some $\delta > 0$ depending on r, s, u and all $\lambda \in \Omega$

$$|\langle \mathrm{Im}(\tilde{P} - \lambda + C_{s,u}\mathrm{Op}(\chi_0^2))u, u\rangle_{L^2}| \geq \frac{C\varepsilon}{2}\|u\|_{L^2}^2 - C'\|u\|_{H^{-\delta}}^2.$$

A similar estimate holds for the adjoint of $\tilde{P}$, which implies by standard argument that $\tilde{P} - \lambda + C_{s,u}\mathrm{Op}(\chi_0^2)$ is Fredholm on $\mathcal{H}^{s,u}$, and using ellipticity of X on $\mathrm{supp}(\chi_0)$, this also implies that $\tilde{P} - \lambda$ is Fredholm on $\mathcal{H}^{s,u}$; see again (see again [FS11, Section 3.2,3.3] for details). The fact that it has index 0 is easily shown by taking $\mathrm{Re}(\lambda) \gg 1$ large, where $\tilde{P} - \lambda$ is invertible. □

Appendix A. Faure-Sjöstrand spaces in finite regularity

Yannick Guedes Bonthonneau

Since the start of the XXIth century, anisotropic Banach spaces of distributions are a main stay of the study of dynamics and spectral theory of Anosov flows. They are used to define and study the so-called Pollicott-Ruelle resonances, whose definition we will recall. Such spaces can be constructed with several different techniques, and we refer to [Bal17] for a glimpse of the mathematical landscape. One particular construction originated in the article [FS11], presented such spaces in the form

$$\mathcal{H}_G(M) := e^{\mathrm{Op}(G)}L^2(M).$$

Here, G is a symbol in a log class, suitably chosen. This kind of space is particularly amenable to the use of microlocal techiques. As can be expected, such techiques are more effective when dealing with smooth systems. Indeed, the Faure-Sjöstrand construction has so far only been used in the case that the flow is C^∞. However, since its inception, the theory of Anosov flows is intended to deal with flows of finite regularity. Our purpose here is to show that, just as several other constructions, the Faure-Sjöstrand spaces can be used to deal with C^r Anosov flows, provide $r > 1$.

Given such a vector field X, and a C^{r-1} potentiel V, let us consider

$$s_0 := \inf\{s \in \mathbb{R} \mid X + V - z \text{ is Fredholm on } L^2 \text{ for } \mathrm{Re}\, z \geq s\}.$$

Certainly, the spectrum of $X + V$ acting on L^2 is discrete in $\{z \mid \mathrm{Re}\, z > s_0\}$. It is not quite clear in general how to access the value of s_0. Let us consider

$$s_1 = s_1(X + V) := \limsup_{t\to+\infty} \frac{1}{t} \log \|e^{t(X+V)}\|_{L^2}.$$

This is the abscissa of convergence of the RHS in the formula

$$(X + V - s)^{-1} = \int_0^{+\infty} e^{t(X+V-s)}dt.$$

Certainly, $s_1 \geq s_0$. We will prove

Theorem 4. *Let X be a C^r Anosov flow, and V a C^{r-1} potential, with $r > 1$. Then there exists $\delta > 0$ and G such that P acts on $\mathcal{H}_G$, and has discrete spectrum in $\{s \in \mathbb{C} \mid \operatorname{Re} s > s_1(X+V) - \delta\}$.*

More precisely, if λ_u (resp. $-\lambda_s$) is the slowest positive (resp. negative) Lyapunov exponent of the flow, we can choose G so that δ is arbitrarily close to

$$(r-1)\frac{\lambda_u \lambda_s}{\lambda_u + \lambda_s}.$$

When the flow is C^∞, δ can be chosen arbitrarily large. In the proof, we will assume that the reader is familiar with the arguments of [FS11] and [DZ16].

Proof. Using some Fredholm theoretic arguments, we have to find G satisfying two requisites. First, to have a non-empty resolvant set, we need that

$$(e^{tP})_{t\in\mathbb{R}^+} \text{ is a strongly continuous semi-group on } \mathcal{H}_G. \tag{A.1}$$

Second, we need to have some larger space $\mathcal{H}'_G$, with a compact injection $\mathcal{H}_G \hookrightarrow \mathcal{H}'_G$, so that for $\operatorname{Re} s > s_1 - \delta$, and $u \in \mathcal{H}_G$ such that $Pu \in \mathcal{H}_G$,

$$\|u\|_{\mathcal{H}_G} \leq C\|(P-s)u\|_{\mathcal{H}_G} + \|u\|_{\mathcal{H}'_G}. \tag{A.2}$$

Let us recall what is meant by "G is a log symbol": we assume that $G = m(x,\xi)\log|\xi|$ mod $|\xi|^{-1}$ when is ξ is large, where m is an order 0 symbol, that we have to construct — indeed, $\mathcal{H}_G$ only depends on m, and not on the lower order asymptotics of G. It is customary to assume that m is constant in neighbourhoods of $E^*_{u,s,0}$.

The action of P on $\mathcal{H}_G$ is equivalent to the action on L^2 of

$$P_G := e^{-\operatorname{Op}(G)} P e^{\operatorname{Op}(G)}.$$

When P has smooth coefficients, the method of [FS11] relies on analyzing P_G with techniques of microlocal analysis. To mimic their arguments using the paradifferential toolbox of section 2, we need to be able to perturb P by operators bounded from H^{s-r+1} to H^s for $s \in (0, r-1)$. Provided that

$$\max m_+ - \min m_- < r-1,$$

such a perturbation of P yields a perturbation of P_G which maps L^2 to some H^ϵ with $\epsilon > 0$, and is thus compact. It follows that the inevitable remainders appearing in paradifferential construction will not have an effect on the Fredholm-ness or bounded-ness of P acting on $\mathcal{H}_G$.

Our next step is to determine the nature of the operator P_G. Using a Taylor formula, we find

$$\begin{aligned} P_G = P + [P, \operatorname{Op}(G)] + \frac{1}{2}[[P, \operatorname{Op}(G)], \operatorname{Op}(G)] + \dots \\ + \int_0^1 \frac{(1-t)^{\lfloor r\rfloor}}{\lfloor r\rfloor!} e^{-t\operatorname{Op}(G)}[\dots[P, \underbrace{\dots}_{\lfloor r\rfloor+1 \text{ times}}] e^{t\operatorname{Op}(G)} dt. \end{aligned}$$

We use the fact that $[P^{\#}, \mathrm{Op}(G)] \in {}^{r-1}\tilde{\Psi}^{0+}_{1,1}$, and similar statements for higher order brackets (which follow from Proposition 2.9) to deduce that

$$P_G = P + [P^{\#}, \mathrm{Op}(G)] + R,$$

where R is a operator compact on L^2 (mapping L^2 to some H^ϵ with $\epsilon > 0$).

To satisfy (A.1), we compute

$$\sigma(\mathrm{Re}\, P^{\#} + [P^{\#}, \mathrm{Op}(G)]) = -\frac{1}{2}\mathrm{div} X + \mathrm{Re}\, V + H_X G.$$

Very much as in the smooth case, using the Gårding inequality, we deduce that (A.1) is satisfied if

$$H_X m \leq 0 \text{ everywhere.}$$

(this implies that $\mathrm{Re}\, P_G \leq C$ for some constant $C > 0$).

Next, to obtain (A.2), we can use the proof of [DZ16] (in particular Proposition 3.4 therein). It relies on using the elliptic parametrix of Proposition 2.10, and the propagation estimates of section 2.3. We will not go into more details, except that the crucial point in the proof will be that estimate (A.2) holds for s when there exists $T > 0$ such that for $|\xi|$ large enough,

$$\frac{1}{T}\int_0^T \sigma(\mathrm{Re}\, P^{\#} + [P^{\#}, \mathrm{Op}(G)]) \circ \Phi_t \, dt < \mathrm{Re}\, s.$$

(this is what will appear in the conditions of "eventual positivity/negativity" at the source and sink.)

Using the change of variable formula, we can compute that

$$s_1(X+V) = \sup_x \limsup_T \frac{1}{T}\int_0^T \left(\mathrm{Re}\, V - \frac{1}{2}\mathrm{div} X\right) \circ \varphi_t \, dt.$$

The proof of the theorem is thus complete, if we can build a smooth function m such that

$$-\delta := \sup_x \limsup_{|\xi|\to\infty \text{ in a neighbourhood of } E_u^* \oplus E_s^*} \limsup_{T\to+\infty} \frac{1}{T}\int_0^T H_X G \circ \Phi_t \, dt < 0. \tag{A.3}$$

Since m is assumed to be constant near E_u^* and E_s^*, we denote these values by $-m_u < 0$ and $m_s > 0$. The assumption that $H_X m \leq 0$ implies that $-m_u \leq m \leq m_s$, and thus we need to ensure that $m_u + m_s < r - 1$.

Some elementary computations then show that if we can find such an m, the value of δ is given by

$$\delta = -\max\left(-m_s\lambda_s^*, -m_u\lambda_u^*\right),$$

where λ_s^* is the slowest contraction rate on E_s^*, and λ_u^* the slowest expansion rate on E_u^*. By duality, $\lambda_s^* = \lambda_u$ and $\lambda_u^* = \lambda_s$, the slowest exponents of dilatation and contraction respectively on E_u and E_s. Certainly, if m satisfies (A.3), then so does $am + b$ if $a > 0$. In particular, optimizing under the constraint $m_u + m_s < r - 1$, we deduce that we can get δ arbitrarily close to the upper bound

$$(r-1)\frac{\lambda_u\lambda_s}{\lambda_u + \lambda_s}.$$

Having found the criteria that m has to satisfy, let us determine one such function. This is the only delicate point. Indeed, the usual construction of m uses the flow of H_X itself on ∂T^*M, which is here only C^{r-1}.

For this, we can use the perturbative argument of [GB18] (this will yield a function with $m_u = m_s = 1$, which we can then rescale to satisfy $m_u + m_s < r - 1$). For $\eta_0 > 0$ we can choose X_0 a C^∞ vector field, so that $\|X - X_0\|_{C^r} < \eta_0$. According to Lemma 3 in [GB18], there exists $\eta > 0$ and $m \in C^\infty(\partial T^*M)$ such that for all C^1 vector fields Y such that $\|Y - X_0\| < \eta$, m is a suitable escape function for Y. We will be done if we can prove that $\eta > \eta_0$. For this, it suffices to prove that the η of Lemma 3 can be chosen locally uniformly *in C^1 topology*.

Inspecting the proof, we find that the η depends on X_0 through its C^1 norm, a lower bound on the angle between its stable and unstable directions, and its hyperbolicity constants. All these quantities can be controlled uniformly in small C^1 open sets of Anosov vector fields, and this concludes the proof. □

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Email address: colin.guillarmou@math.u-psud.fr

Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France

Email address: tdepoyferre@msri.org

MSRI Berkeley, 17 Gauss Way, Berkeley, CA 94720
Email address: bonthonneau@math.univ-paris13.fr

LAGA, Institut Galilée, 99 avenue Jean Baptiste clément, 93430 Villetaneuse, France.